PROPAGATION OF HARMONIC WAVES IN LAYERED TRANSVERSELY ISOTROPIC SHELLS

By

HENRY EVANS KECK

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
1968

DEDICATION

From women's eyes this doctrine I derive:
They sparkle still the right Promethean fire;
They are the books, the arts, the academes,
That show, contain, and nourish all the world;
Else none at all in aught proves excellent.

Love's Labour's Lost, IV, iii
Shakespeare

These words, spoken for me by Shakespeare, reveal my true feelings as I dedicate this work to Jeanne, my wife.

H.E.K.

ACKNOWLEDGMENTS

The author thanks, with sincere appreciation, Dr. A. E. Armenàkas, dissertation chairmen of his supervisory committee, for guidance and tutelage in the commission of this work. The debt here can not be estimated. Additional thanks, for continuing interest and concern in the author's progress, are given to Dr. I. K. Ebcioglu, committee chairman, Dr. G. E. Nevills, Jr., Dr. S. Y. Lu, Dr. K. Yao, and Dr. L. Janos, members of the supervisory committee.

A separate word of appreciation is proffered to Dr. M. A. Eisenberg for the many and thoughtful discussions, given in the spontaneity of academic interest.

The author is further indebted to the United States Air Force, in a manifold manner: for providing the chance to pursue graduate studies; for providing the assistance of the personnel at the Flight Dynamics Laboratory and the Digital Computation Division, Wright-Patterson Air Force Base, Ohio, without whose aid numerical results could not have been obtained.

TABLE OF CONTENTS

		Page
ACKNOWLEDGMENTS		ii
LIST OF FIGURES		vi
KEY TO SYMBOLS		vii
ABSTRACT		xi
CHAPTER		
I.		1
II.		10
	1. Basic Equations	10
	2. Displacement Equations of Motion	16
III.		20
	1. On the Use of Displacement Potential Functions to Solve the Equations of Motion	20
	2. A Completeness and Representation Theorem for General Harmonic Waves in Transversely Isotropic Bodies	27
	3. Alteration of the Representation for Two Special Cases	35
	4. Specification of $\varphi(r,\theta)$ and $f(r,a)$	39
IV.		43
	1. Analysis of the Radial Wave Number s, and f(r)	43
	2. Analysis of the Radial Wave Numbers p and q, and $\varphi(r)$	48
V.		68
	1. Preliminaries of the Problem and Nondimensionalization of Equations	68
	2. Derivation of the Frequency Equation	75
	3. The Ω-ξ Plane	85
	4. Turning Point Line Expressions for the C_{ij}	92
	5. Complex Zone Line Expressions for the C_{ij}	97

		Page
6.	The Frequency Equation Representation in Complex Zones	98
VI.		104
1.	Frequency Equation for Three Layered, Traction Free, Isotropic Cylinder	104
2.	Frequency Equation for Waves of Infinite Length	108
3.	Frequency Equation for Axisymmetric Traveling Waves	113
4.	Frequency Equation for Axisymmetric Waves of Infinite Length	117
5.	Frequency Equation for One and Two Layered Shells	121
6.	Layered Cylinders with Clamped-Free and Clamped-Clamped Boundary Conditions	124
7.	Frequency Equations for Cylinders with Clamped-Free and Clamped-Clamped Boundary Conditions When $\zeta = n = 0$	132
8.	Frequency Equation for Cylinders with Very Thin Outer Layers	135
9.	N-Layered Rods and Cylindrical Shells	139
VII.		141
1.	The Frequency Equation in Sector 1 for $\zeta \to \infty$	144
	A. The Equation $D_6^{(i)} = 0$ $(i = 1,3)$	148
	B. The Equation $D_7^{(i)} = 0$ $(i = 1,3)$	151
	C. Summary of Sector 1 Limits	159
2.	The Frequency Equation in Sector 2 for $\zeta \to \infty$	160
	A. Sandwich Shells - Sector 2	161
	B. Core Layered Shells - Sector 2	170
	C. General Shells - Sector 2	173
3.	The Frequency Equation for Higher Sectors	174
4.	Summary of the Asymptotic Behavior of the Frequency Equation When $\zeta \gg 1$	176
VIII.		180
1.	The Frequency Equation for Infinite, Three Layered Plates	180

TABLE OF CONTENTS (Continued)

		Page
IX.		187
	1. Method of the Numerical Analysis	187
	2. Discussion of the Frequency Spectra	193
	3. Discussion of the Vibration Displacement Distributions	200
BIBLIOGRAPHY		216
BIOGRAPHICAL SKETCH		226

LIST OF FIGURES

Figure		Page
1.	c_{11} versus $\rho\omega^2/k^2\big\|_i$ $(i = 1,2,3)$	54
2.	Inequality (4:24)	57
3.	Radial Wave Numbers versus $\rho\omega^2/k^2$	60
4.	Radial Wave Numbers versus $\rho\omega^2/k^2$	62
5.	Cross-Section of Cylinder	69
6.	Ω-ζ Plane with Zone Lines	91
7.	The $\bar{\mu}$ versus δ^{02} Plane	157
8.	Plate Cross-Section	181
9.	Frequency Spectrum: Homogen., H/R = 1.0	203
10.	Frequency Spectrum: Homogen. (cont), H/R = 1.0	204
11.	Frequency Spectrum: Case I, H/R = 1.0	205
12.	Frequency Spectrum: Case I (cont), H/R = 1.0	206
13.	Frequency Spectrum: Case II, H/R = 1.0	207
14.	Frequency Spectrum: Case II (cont), H/R = 1.0	208
15.	Frequency Spectrum: Homogen., H/R = 0.1	209
16.	Frequency Spectrum: Homogen. (cont), H/R = 0.1	210
17.	Frequency Spectrum: Case I, H/R = 0.1	211
18.	Frequency Spectrum: Case I (cont), H/R = 0.1	212
19.	Frequency Spectrum: Case II, H/R = 0.1	213
20.	Frequency Spectrum: Case II (cont), H/R = 0.1	214
21.	Longitudinal-Shear Vibration Distributions	215
22.	Radial Plane-Strain Vibration Distributions	216
23.	Circumferential-Shear Vibration Distributions	217

KEY TO SYMBOLS

Though the usage of symbols in this work is, in the main, consistent with modern engineering usage, and though each symbol, when introduced, is defined, the following listing should prove of use when the reader is tempted to begin his study in the middle of the text, rather than the beginning. All symbols listed below may occur unsuperscripted (in Chapters I through IV) and may in subsequent chapters bear the superscript "(i)", where i = 1, 2, or 3. This marking denotes the material layer of the three layered shell to which the symbol is, at that moment, ascribed, where "(1)" is for the innermost layer, and "(3)" is for the outer.

τ_{ij}	the i,j component of stress in an orthogonal, cartesian coordinate frame
ϵ_{ij}	the i,j component of strain in an orthogonal, cartesian coordinate frame
F_i	the i component of body force
C_{ijkl}	fourth order tensor of elastic stiffnesses for a general anisotropy
τ_i	the i component of stress in contracted notation, defined by (2:4)
ϵ_i	the i component of strain in contracted notation, defined by (2:4)
c_{ik}	C_{ijkl} redefined corresponding to the contracted notation of (2:4)

s_{ij}	elastic compliances, defined by equation (2:9)
u_i	the i component of displacement
ρ	density
W	strain energy density, defined initially on page 11
λ, μ	Lame isotropic elastic constants
ν, E	Poisson's ratio, Young's modulus
r, θ, z	cylindrical coordinates
x, y, z	cartesian coordinates
$\bar{r}$	nondimensional radial coordinate, defined by (5:22)
$\Box^2$	wave operator, defined by (3:3)
∇^2	Laplace operator
∇_1^2	Laplace operator in r and θ, defined by (3:11a)
ϕ, ψ, ψ_i	various potential functions used generally in Section 1 of Chapter III
n	number of circumferential waves, defined initially by (3:12)
k	wave number, defined following (3:12)
$\bar{\lambda}$	wavelength, defined following (3:12)
ω	natural frequency, defined following (3:12)
v	phase velocity, defined following (3:12)
c_1, c_2	wave speeds in infinite isotropic media, defined by (3:4b)
v_g	group velocity, defined by (5:39)
Ω	nondimensional frequency, defined by (5:2)
$\omega_c^{(1)}$	frequency of first antisymmetric thickness shear mode, defined by (5:1)
$\bar{\gamma}$	nondimensional wave number, defined by (5:3)

$\mathcal{Z}$	nondimensional phase velocity, defined by (5:8)
v_s, v_d	wave velocities, defined by (5:4) and (5:6)
$b, \bar{b}$	nondimensional wave velocity ratios, defined by (5:5) and (5:7)
d_i	$i = 1,2, \ldots, 6$ nondimensional elastic constants, defined by (5:10)
p, q, s	radial wave numbers, defined by (3:17b) and (3:17c)
α, β, γ	factor of the radial wave numbers p, q, s, respectively, defined by (4:3), (4:37), and (4:38)
ϵ_i	$i = 1,2,3$ corresponding to the shell layers, sign factors defined by (4:2), (4:34), and (4:35)
ρ, η, ξ	nondimensional radial wave numbers, defined by (5:12) and (5:18)
α, β, ξ	nondimensional factors of the radial wave numbers, defined by (5:16), (5:17), and (5:20b)
$\frac{\omega^2}{k^2}\Big\|_{ki}$	$i = 1,2,3$ corresponding to critical values of the given ratio, defined in (4:12) and (4:13), for each layer of the shell
$\mathcal{Z}\|_{k_1}, \mathcal{Z}\|_{k_2}$	nondimensional complex zone line numbers, defined by (5:15)
J_n, Y_n	regular Bessel functions of order n
I_n, K_n	modified Bessel functions of order n
Z_n	regular or modified Bessel function of the first kind, of order n, defined by (4:40)
W_n	regular or modified Bessel function of the second kind, of order n, defined by (4:40)
A_i, B_i	$i = 1,2,3$ corresponding to specific displacement potentials, integration constants in the general solution
$\mathcal{Z}_{n_1} \mathcal{W}_n$	$n = 0$ or 1, special representation of plate solution, defined by (8:7) and (8:9)
$\det\|( )\|$	used to denote the determinate of ( )

D_i, E_i, H_i	i an integer, used to denote special determinates
C_{ij}	elements of the frequency equation, defined initially in (5:37)
$\bar{C}_{ij}$	complex elements of the frequency equation, defined by (5:50)
m	slope of a ray emanating from origin of the $\Omega\,\xi$ plane, defined by (7:1)
m_R	slope of ray on frequency spectrum corresponding to Rayleigh wave velocity, defined by (7:12)
$\bar{\mu}$	ratio of $d_6^{(i)}$'s of adjacent layers, defined by (7:14)
$\varphi^{(i)}, \psi^{(i)}$	angular argument of asymptotic expressions for regular Bessel functions of large argument, defined by (7:27). Here "(i)" refers to material layer of the shell
h	shell layer thickness, defined in Figure 5
R	mean radius of the middle layer of the shell, defined in Figure 5
$(\,)^*$	complex conjugate designation
R, Re	denote real part of complex argument
I, Im	denote imaginary part of complex argument
i	square root of minus one

Abatract of Dissertation Presented to the Graduate Council in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

PROPAGATION OF HARMONIC WAVES IN LAYERED TRANSVERSELY ISOTROPIC SHELLS

By

Henry E. Keck

August, 1968

Chairman: Dr. Ibrahim K. Ebcioglu
Co-chairman: Dr. Anthony E. Armenàkas
Major Department: Engineering Science and Mechanics

The propagation of harmonic waves in infinitely long, right circular cylindrical shells and rods is treated on the basis of the three-dimensional theory of elasticity. The shell or rod is assumed made of three concentric, transversely isotropic cylinders, each of different material, bonded perfectly at their interfaces. The displacement field in each cylinder is represented in terms of potential functions and these are subsequently used to uncouple and solve the Navier equations of motion. The boundary and interface conditions of the cylinder are used to specify the integration constants of the solution. The condition of nontrivial solutions for these constants produces an 18X18 determinant, representing a transcendental relationship between the frequency, wavelength of the traveling waves, and the number of circumferential waves.

Thia frequency equation yields all posaible modes of propagation in the cylinder. The caaea of infinite wavelength, axiaymmetric waves, and axisymmetric waves of infinite wavelength are treated for traction free and rigidly clamped cylindera and roda. The frequency equatioe ia developed for the case of iaotropic layers. The case of very thin outer layers, in the framework of the exact aolution, ia conaidered. An eaymptotic analyaia of the frequency equatioe ia conducted for the caae of very short wavelength traveling waves. The aolution for the cylinder ia treated in the limit of infinite radiua and the corresponding aolution and concomitant wave equations are developed for e transverealy iaotropic, three-layered plate. Finally, e numerical analyaia, baaed on e computer program capable of every aapect of the above problem, ia presented for exiaymmetric wavea in iaotropic layered cylinders.

CHAPTER 1

The mechanics of wave motion of all types has long been a topic of interest to physicists and scientists. At the macroscopic level perhaps the theories are furthest advanced in connection with the electromagnetic and elastic field theories. In the elastic domain the desire for knowledge is far surpassed by the need for practical, applicable results. We can not know enough about wave motion to progress beyond application of this knowledge in the engineering world of our age.

To obtain a complete understanding of the wave patterns and their ensuing stress and strain fields in any elastic body, an investigation must be phrased in terms of a general elastic body - representative of the particular case of interest. Moreover, a source must be hypothesized in the body, exciting a generality of local time varying strain fields. From this, the formulation of the problem must be framed in an acceptable mathematical model. Appropriate boundary conditions and differential equations arise in this process which, with the exception of almost trivial cases, resist the most persistent efforts of solution.

Numerous avenues of escape exist from this dilemma. One may use a more crude mathematical model, or barring this (but which is almost the same), may linearize and otherwise modify

the equations of the problem to render them soluble. Another alternative is to modify the elastic body, specifying, for instance, a semi-infinite medium, an infinite plate, a cylinder of infinite axial length, etc. For such a specification some of the boundary conditions are simplified and the work involved in manipulating the rigorous equations may be rendered more tractable. Further reduction in the complications of the general problem is obtained if initial conditions for the time dependent part of the solution are in some way obviated.

With the source located at infinity in the axial direction of an infinite elastic cylinder, it is clear that all those forms of wave motion whose amplitudes are attenuated with distance will not be observed unless the patterns of reflections in some way reinforce each other. If wave motion is observed, it is then natural to assume that the amplitudes of passing waves do not change; that is, it is of little consequence at what point along the elastic body one picks to observe the wave motion - as long as the source is an infinity away. It is also convenient to pick a source which will generate infinite trains of sinusoidally varying traveling waves. This is hardly the compromise it seems, as can be seen from [1,2][1]. The method of Fourier analysis, using these harmonic wave results, makes it possible to study more complicated wave shapes. Thus, when the elastic

---

[1]Numbers in square brackets refer to references listed in the Bibliography.

properties are described for the body in question, and when the boundary conditions on the bounded surfaces are decided, there remains only the task of choosing a mathematical model which will yield to some method of solution.

Much work was done in the early 1800's by Poisson, Navier, Stokes, Rayleigh, and Lagrange [3,4,5, and 6]. These early efforts were based on a finite number of modes of wave motion in semi-infinite bodies, plates, and bars. These first "approximate" solutions were valid only in the low frequency, long wavelength range. Solutions, based on linear elastic theory, of wave problems in infinite plates by Lamb [7] and infinitely long cylinders by Pochhammer [8] and Chree [9] were presented at the turn of the century. Because of the complexity of the resulting frequency equations, they were not evaluated numerically until recently. However, their numerical evaluation is difficult even using electronic computers, and for this reason several modern approximate theories have been established. Mirsky and Herrmann [10,11] produced a five mode Timoshenko-type theory containing the usual membrane and bending effects plus the influence of transverse shear deformation and rotatory inertia. A three mode theory, developed by McNiven, Shah, and Seckman [12,13], containing the framework for inclusion of higher modes, utilized correction factors which matched the approximate mode curves to the exact theory mode curves at two places along each exact curve. This theory, which is for hollow cylinders, is

an extension of a rod theory developed by Mindlin and McNiven [14]. Naghdi and Cooper [15,16] provided a three mode approximate theory for thin shells. Two sets of governing differential equations were presented based generally on the inclusion and exclusion, respectively, of small magnitude terms involving the shell thickness and middle surface strains. The first set of these equations reduces to Love's first approximation [17] and the second set to Donnell's equations [18], when transverse shear deformation and rotatory inertia are neglected.

The advent of computers has brought about a thorough examination of the Pochhammer-Chree frequency equation (that is, the exact solution). Mindlin, Onoe, and McNiven [19] and Mindlin and Fao [20] have fully described the behavior of the branches of the frequency equation for real, imaginary, and complex propagation constant (wave number). Gazis [21,22] extended the exact theory for traveling waves and vibrations to thick walled hollow cylinders, and Greenspon [23] independently presented the vibration theory for cylinders.

The dispersive characteristics of axially symmetric waves in infinitely long, two layered, elastic rods for the case of asymmetric nontorsional waves were investigated on the basis of the linear theory of elasticity by McNiven, Sackman, and Shah [24] and of axisymmetric torsional waves by Armenàkas [25]. Baltrukonis, et al. [26] solved the problem of an elastic cylinder rigidly encased, and later

treated longitudinal shear vibrations in shells composed of two elastic layers [27] and plane strain vibrations in composite shells with an elastic casing and incompressive inner layer [28]. Armenàkas [29] presented a unified theory for symmetric and antisymmetric harmonic waves in two layered cylinders with traction free surfaces. Armenàkas and Keck [30] presented the asymptotic behavior, for very large propagation constant, of the frequency equation for traveling harmonic waves in composite two layered, traction free, elastic rods. Jones and Whittier [31] have presented stress and displacement field distributions on the basis of linear elastic theory for the case of axisymmetric wave propagation in two layered, traction free, elastic cylinders.

The development of the theory - as applied to anisotropic elastic bodies - has received little attention prior to recent times. This was partly due to the increased difficulty of the mathematical aspects of the problem, and partly because there was little need, other than pedagogical, for such developments.

The development of the exact linear theory as applied to tubes and cylinders of anisotropic materials was initiated by Chree [32] in 1890 with a discussion of the propagation of harmonic waves in transversely isotropic[1] rods of infinite axial length. His solution, however, was based on

---

[1]Transverse isotropy is characterized by any material which possesses a plane in which the elastic properties are the same in all directions, and the elastic properties in the normal direction to this plane are different.

approximate methods and has little value outside the low frequency, long wavelength region. In 1954 Morse [33] developed the exact solution for compressional waves traveling in solid rods of a transversely isotropic material. In the 1960's Mirsky [34,35] produced an exact solution for axisymmetric vibrations in orthotropic[1] cylinders. This presentation utilized the method of Frobenius to solve the equations of motion and, consequently, represented a significant departure from the usual method of solution of problems of this kind by use of potentials. Subsequently, Mirsky [36] presented an approximate theory for thick walled orthotropic cylinders of the same nature and degree of accuracy as the Mirsky-Herrmann isotropic cylindar approximate theory. Two years following the vibration solution by Mirsky, Nowinski [37] published a paper giving the exact solution for traveling waves in orthotropic rods. He also used the method of Frobenius to deal with the equations of motion. Mirsky published a paper [38] in 1965 dealing with the completeness of solutions for anisotropic elastic problems when the displacements are represented in terms of potential functions, and a two part paper [39,40] providing the solution, in terms of displacement potential functions, for traveling waves in transversely isotropic rods and cylinders.

The exact solution of the traveling wave problem in

---

[1]Orthotropy is the name given a material possessing the property of three mutually orthogonal planes of elastic symmetry. Such a material is characterized by nine independent elastic constants.

composite anisotropic rods and cylinders, that is, in rods and cylinders with two or more bonded layers each of a different material, has not been established. Yu [41] has presented an approximate theory for vibrations in isotropic sandwich elastic cylindrical shells and Chu [42] for honeycomb core sandwich cylinders. Both of these theories include the effects of rotatory inertia and transverse shear deformation. The validity of such theories, however, must be determined experimentally or by comparison with solutions established on the basis of the three-dimensional theory of elasticity.

Use of monocrystalline materials (for example, quartz crystals in electronics) has created an additional need for wave theory in the ultrasonic frequency range applicable to single crystal plates of various anisotropies. Furthermore, plastics, fiber reinforced shells with hexagonal elastic symmetry or near hexagonal symmetry [43], and recently introduced engineering materials in the aerospace industry, such as graphite and Berellium, all present differing requirements for establishing the transverse isotropic theory. Moreover, although polycrystalline materials exhibit little tendency toward anisotropic properties under ordinary conditions [44], when rolled into thin sheets metals such as steel exhibit a mild transverse isotropy [45]. Sheet Berellium [46] likewise exhibits this property, but to a more distinct degree.

The theory contained herein, which is established for three layered cylinders and plates, aside from being a

contribution to the exact elastic theory, will provide the required analytical basis for many of the problem areas cited above.

In this investigation the propagation of an infinite train of harmonic waves in infinitely long, right circular, cylindrical shells and rods will be treated on the basis of the three-dimensional theory of elasticity. The shell or rod will be assumed made of three concentric, transversely isotropic cylinders, each of different materials, bonded perfectly at their interfaces. The displacement field in each cylinder will be represented in terms of potential functions and these will subsequently be used to uncouple and solve the Navier equations of motion [47]. The boundary and interface conditions of the cylinder will be used to specify the integration constants of the solution. Nontrivial solutions for these constants produce an 18X18 determinant which is variously referred to as the frequency or dispersion equation. This equation, representing a transcendental relationship between the frequency and wavelength of the traveling waves, and number of circumferential waves, yields all possible modes of propagation in the cylinder. It will be shown, as in the case of simple isotropic shells, that the frequency equation degenerates into simpler cases for axially symmetric waves (torsional and nontorsional) and for waves having infinite axial wavelength (longitudinal shear and plane strain). Asymptotic expressions will be presented for the behavior of the frequency equation when the wave number is very large.

The limiting case, as the radius of the cylinder becomes infinite, will be treated, and the frequency equation for three layered plates established.

A computer program will be written for the general problem outlined above. Because of limited availability of machine time numerical results will be presented only for the case of axisymmetric waves traveling in shells composed of isotropic layers.

CHAPTER II

1. Basic Equations

The principle of conservation of linear momentum of an element dV contained in a body D bounded by the surface S gives [47,48]

$$\tau_{ij,i} + F_j = \rho \ddot{u}_j \tag{2:1}$$

where ρ is the mass density per unit volume of D, τ_{ij} are the components of the stress tensor from the linear theory of deformable continua, and F_j are the components of the body forces. Differentiation with respect to time and space have been indicated, respectively, by the "dot" and "comma" conventions. A repeated subscript indicates summation over all possible values of the indices $(i,j = 1,2,3)$. The components of displacement u_j, are related to the components of strain of a linearly deformable continuum by the relation

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2:2}$$

The stress tensor will be assumed related to the strain tensor by the following constitutive relation:

$$\tau_{ij} = C_{ijkl}\,\epsilon_{kl}, \quad (i,j,k,l = 1,2,3) \tag{2:3}$$

Since the stress and strain tensors are symmetric, it follows that

$$C_{ijkl} = C_{jikl} = C_{ijlk} .$$

This expression reduces the number of independent constants in the tensor, C_{ijkl}, from 81 to 36.

If the strain components in a unit element of the body D are increased from ϵ_{ij} to $\epsilon_{ij} + d\epsilon_{ij}$, the work done is [44]

$$dW = \tau_{ij}\, d\epsilon_{ij} ,$$

and, using (2:3), this gives

$$dW = C_{ijkl}\epsilon_{kl} d\epsilon_{ij} .$$

Consequently,

$$\frac{\partial W}{\partial \epsilon_{ij}} = C_{ijkl}\epsilon_{kl} ,$$

and

$$\frac{\partial^2 W}{\partial \epsilon_{ij}\partial \epsilon_{kl}} = C_{ijkl} .$$

Alternatively, interchanging the dummy subscripts results in

$$\frac{\partial W}{\partial \epsilon_{kl}} = C_{klij}\,\epsilon_{ij}$$

or

$$\frac{\partial^2 W}{\partial \epsilon_{kl}\partial \epsilon_{ij}} = C_{klij} .$$

Therefore, since the order of differentiation is immaterial [17], it follows that

$$C_{ijkl} = C_{klij} .$$

This last relation reduces the number of independent elastic constants from 36 to 21, which is the number of constants

for a general anisotropy.

To avoid dealing with double sums and large groups of subscripts the following notation is introduced

$$\tau_{11}=\tau_1,\ \tau_{22}=\tau_2,\ \tau_{33}=\tau_3,\ \tau_{23}=\tau_4,\ \tau_{31}=\tau_5,\ \tau_{12}=\tau_6,$$
$$\varepsilon_{11}=e_1,\ \varepsilon_{22}=e_2,\ \varepsilon_{33}=e_3,\ 2\varepsilon_{23}=e_4,\ 2\varepsilon_{31}=e_5,\ 2\varepsilon_{12}=e_6. \qquad (2:4)$$

In terms of (2:4), equations (2:3) become

$$\tau_i = c_{ij}e_j \quad (i,j=1,2,\ldots,6) \qquad (2:5)$$

The constitutive relation (2:5), which is the generalized Hooke's Law of the linear small deformation elastic theory, can only be further simplified, if the mechanical response of the material considered is symmetric with respect to axes or planes in the solid [17,48], etc. If the elastic properties are symmetric with respect to three mutually orthogonal planes, the number of independent elastic constants is reduced to nine and (2:5) yields the six equations of orthotropy

$$\tau_i = c_{ij}e_j \quad (i=1,2,3;\ j=1,2,3),$$
$$\tau_k = c_{kk}e_k \quad (k=4,5,6,\ \text{no sum}). \qquad (2:6)$$

If the elastic properties of the solid are symmetric with regard to an axis of symmetry, z, the nine independent constants in (2:6) are reduced to five and the solid is termed transversely isotropic. For this case equation (2:5), written in expanded form, given

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3,$$
$$\tau_2 = c_{12}e_1 + c_{11}e_2 + c_{13}e_3, \qquad (2:7)$$

$$\tau_3 = c_{13}e_1 + c_{13}e_2 + c_{33}e_3\,,$$
$$\tau_4 = c_{44}e_4\,, \quad \tau_5 = c_{44}e_5\,, \quad \tau_6 = c_{66}e_6\,, \qquad (2:7)\ \text{(cont)}$$

where

$$c_{66} = \tfrac{1}{2}(c_{11} - c_{12}).$$

The fact that the strain energy of an elastic solid increases through positive values, from zero in the strain free state, imposes restrictions on the range of values to which the elastic constants may obtain. These limitations are brought out most easily by noting that the strain energy density function is a positive definite quadratic form. Hence, in the usual manner of establishing the criteria for positive definiteness of a quadratic form [49], each of the inequalities below, for p = 0 to 5, must hold

$$\begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1,p+1} \\ c_{21} & c_{22} & \cdots & c_{2,p+1} \\ \cdots & & & \cdots \\ c_{p+1,1} & c_{p+1,2} & & c_{p+1,p+1} \end{vmatrix} > 0\,.$$

Hence, for transverse isotropy in particular

$$\begin{matrix} c_{11} > 0 \\ c_{33} > 0 \\ c_{44} > 0 \end{matrix}\,, \quad \begin{matrix} c_{11}^2 - c_{12}^2 > 0 \\ c_{11}c_{33} - c_{13}^2 > 0 \end{matrix}\,, \quad c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2 > 0\,, \qquad (2:8)$$

are the limitations imposed on the elastic constants by positive definite strain energy density.

The elastic stiffnesses c_{ij} in formulas (2:6) and (2:7) can be related to engineering moduli and Poisson's ratios in

the following fashion. The determinant of the elastic stiffnesses physically can not be zero. Thus, equations (2:5) can be inverted and the strains expressed as linear combinations of the stresses

$$e_i = s_{ij}\tau_j \quad (i,j = 1,2,\ldots,6), \qquad (a)$$

where (2:9)

$$\det|s_{ij}|\ \det|c_{ij}| = 1. \qquad (b)$$

The s_{ij} are termed elastic compliances and bear the following physical significance [44]:

a) s_{qq} (q = 1,2, or 3) relates an extensional stress to an extensional strain both in the same direction, and $s_{qq} = 1/E_q$;

b) s_{qr} (q ≠ r; q,r = 1,2, or 3) relates an extensional strain to a perpendicular extensional stress, and

$$s_{qr} = -\nu_{rq}/E_r = -\nu_{qr}/E_q;$$

c) s_{qr} (related to s_{ijj}) (q = 1,2, or 3; r = 4,5, or 6) relates an extensional strain to a shear stress and vice versa, both in the same plane,

s_{qr} (related to s_{iij}) (q = 1,2, or 3; r = 4,5, or 6) relates an extensional strain to a shear stress in a perpendicular plane and vice versa;

d) s_{qq} (q = 4,5, or 6) relates a shear strain to a shear stress in the same plane, and

$$s_{qq} = 1/G_q,$$

e) s_{qr} (q ≠ r; q,r = 4,5, or 6) relates a shear strain to a shear stress in a perpendicular plane.

Because of the analytical similarity of equations (2:9) and (2:5), the requirements used to reduce (2:5) to the orthotropic or transverse isotropic cases will reduce (2:9) to the same configuration, except, taking note of (2:4),

$$s_{66} = 2(s_{11} - s_{12})$$

The c_{ij} can be related to engineering parameters E_q, G_q only through the compliances. From (2:9b) the relation

$$c_{ij} = \Delta s_{ij} / \det |s_{ij}| \quad , \tag{2:10}$$

where Δs_{ij} is the minor determinant associated with s_{ij}, is easily obtained. Consequently, the following representation for the c_{ij} in terms of the s_{ij} and the engineering constants results for the transverse isotropic case.

$$c_{11} = \frac{s_{11}s_{33} - s_{13}^2}{(s_{11} - s_{12})X} = \frac{E_1(1 - \nu_{13}\nu_{31})}{(1 + \nu_{12})(1 - \nu_{12} - 2\nu_{13}\nu_{31})} \quad ,$$

$$c_{12} = \frac{s_{13}^2 - s_{12}s_{33}}{(s_{11} - s_{12})X} = \frac{E_1(\nu_{12} + \nu_{13}\nu_{31})}{(1 + \nu_{12})(1 - \nu_{12} - 2\nu_{13}\nu_{31})} \quad ,$$

$$c_{13} = \frac{-s_{13}}{X} = \frac{E_3\nu_{13}}{1 - \nu_{12} - 2\nu_{13}\nu_{31}} \quad , \tag{2:11}$$

$$c_{33} = \frac{s_{11} + s_{12}}{X} = \frac{E_3(1 - \nu_{12})}{1 - \nu_{12} - 2\nu_{31}\nu_{13}} \quad ,$$

$$c_{44} = \frac{1}{s_{44}} = G_4 \quad , \quad c_{66} = \frac{1}{s_{66}} = \frac{E_1}{2(1 + \nu_{12})} \quad ,$$

and

$$X = s_{33}(s_{11} + s_{12}) - 2s_{13}^2 = (1 - \nu_{12} - 2\nu_{13}\nu_{31})/E_1E_3 \, .$$

Similar relations are easily obtained for orthotropy. The ultimate reduction in the number of independent,

elastic constants is obtained when the mechanical properties of the solid are the same in any direction. This is isotropy and is characterized by two independent constants. The stress-strain relations are

$$\begin{aligned} \tau_1 &= c_{11}e_1 + c_{12}e_2 + c_{12}e_3 , \\ \tau_2 &= c_{12}e_1 + c_{11}e_2 + c_{12}e_3 , \\ \tau_3 &= c_{12}e_1 + c_{12}e_2 + c_{11}e_3 , \\ \tau_k &= \tfrac{1}{2}(c_{11} - c_{12})e_k , \quad (k = 4,5,6). \end{aligned} \tag{2:12}$$

The elastic constants in (2:8) may be associated with the Lame coefficients μ and λ, or Young's modulus E and Poisson's ratio ν through the following scheme:

c_{11}	$\lambda + 2\mu$	$E(1-\nu)/(1+\nu)(1-2\nu)$
c_{12}	λ	$E\nu/(1+\nu)(1-2\nu)$
$(c_{11}-c_{12})/2$	μ	$E/2(1+\nu)$
$(c_{11}+2c_{12})(c_{11}-c_{12})/(c_{11}+c_{12})$	$\mu(3\lambda+2\mu)/(\lambda+\mu)$	E
$c_{12}/(c_{11}+c_{12})$	$\lambda/2(\lambda+\mu)$	ν

2. Displacement Equations of Motion

Since the development required in this work must ultimately be phrased in cylindrical coordinates, all derivations will be carried forward in the coordinate frame r, θ, z, where z corresponds to the axial coordinate and r and θ are defined in the usual way.

In cylindrical coordinates the equations of motion may be written explicitly as

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho \frac{\partial^2 u_r}{\partial t^2} ,$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} = \rho \frac{\partial^2 u_\theta}{\partial t^2} ,$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\tau_{rz} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2}$$

Because the stress-strain equations for anisotropy do not yield to a convenient tensorial notation as in the case of isotropy, the development will be continued in an expanded form to arrive at working equations.

The geometric strain-displacement equations are

$$\epsilon_{rr} = e_1 = \frac{\partial u_r}{\partial r} ,$$

$$\epsilon_{\theta\theta} = e_2 = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} ,$$

$$\epsilon_{zz} = e_3 = \frac{\partial u_z}{\partial z} ,$$

$$2\epsilon_{\theta z} = e_4 = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} , \qquad (2:13)$$

$$2\epsilon_{zr} = e_5 = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} ,$$

$$2\epsilon_{r\theta} = e_6 = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} .$$

By substituting equations (2:13) into (2:7) the following stress-displacement equations are obtained

$$\tau_{rr} = \tau_1 = c_{11}\frac{\partial u_r}{\partial r} + c_{12}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + c_{13}\frac{\partial u_z}{\partial z} ,$$

$$\tau_{\theta\theta} = \tau_2 = c_{12}\frac{\partial u_r}{\partial r} + c_{11}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + c_{13}\frac{\partial u_z}{\partial z} ,$$

$$\tau_{zz} = \tau_3 = c_{13}\frac{\partial u_r}{\partial r} + c_{13}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z} , \qquad (2:14)$$

$$\tau_{\theta z} = \tau_4 = c_{44}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) ,$$

$$\tau_{zr} = \tau_5 = c_{44}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) ,$$

$$\tau_{r\theta} = \tau_c - \frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right). \qquad \begin{matrix}(2:14)\\(\text{cont})\end{matrix}$$

Substituting equations (2:14) into the stress equations of motion and rearranging terms, the displacement equations of motion for a transversely isotropic solid with elastic symmetry axis coincident with the z axis are

$$c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) + c_{66}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + c_{44}\frac{\partial^2 u_r}{\partial z^2} + (c_{66}+c_{12})\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r \partial \theta} -$$

$$-(c_{66}+c_{11})\frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} + (c_{13}+c_{44})\frac{\partial u_z}{\partial r \partial z} = \rho\frac{\partial^2 u_r}{\partial t^2}, \qquad \text{(a)}$$

$$(c_{12}+c_{66})\frac{1}{r}\frac{\partial^2 u_r}{\partial r \partial \theta} + (c_{11}+c_{66})\frac{1}{r^2}\frac{\partial u_r}{\partial \theta} + c_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) +$$

$$+\frac{c_{11}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + c_{44}\frac{\partial^2 u_\theta}{\partial z^2} + (c_{13}+c_{44})\frac{1}{r}\frac{\partial^2 u_z}{\partial \theta \partial z} = \rho\frac{\partial^2 u_\theta}{\partial t^2}, \qquad \text{(b)} \quad (2:15)$$

$$(c_{13}+c_{44})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial z}\right) + c_{33}\frac{\partial^2 u_z}{\partial z^2} +$$

$$+c_{44}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}\right) = \rho\frac{\partial^2 u_z}{\partial t^2}. \qquad \text{(c)}$$

In a similar manner, using equations (2:13) in equations (2:6) and subsequently using these equations in the equations of motion, the displacement equations of motion for a cylindrically orthotropic solid, with planes of symmetry coincident with the coordinate planes, are obtained as

$$c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r}\right) + (c_{12}+c_{66})\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r \partial \theta} + (c_{13}+c_{55})\frac{\partial^2 u_z}{\partial r \partial z} + c_{55}\frac{\partial^2 u_r}{\partial z^2} +$$

$$+(c_{13}-c_{23})\frac{1}{r}\frac{\partial u_z}{\partial z} - (c_{22}+c_{66})\frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} - c_{22}\frac{u_r}{r^2} + c_{66}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \rho\frac{\partial^2 u_r}{\partial t^2}, \qquad \text{(a)}$$

$$(c_{12}+c_{66})\frac{1}{r}\frac{\partial^2 u_r}{\partial r \partial \theta} + (c_{23}+c_{44})\frac{1}{r}\frac{\partial^2 u_z}{\partial \theta \partial z} + (c_{22}+c_{66})\frac{1}{r^2}\frac{\partial u_r}{\partial \theta} + \qquad (2:16)$$

$$+c_{22}\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + c_{44}\frac{\partial^2 u_\theta}{\partial z^2} + c_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}u_\theta\right) = \rho\frac{\partial^2 u_\theta}{\partial t^2}, \qquad \text{(b)}$$

$$(c_{13}+c_{55})\frac{\partial^2 u_r}{\partial r\partial z} + (c_{23}+c_{44})\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial z} + (c_{23}+c_{55})\frac{1}{r}\frac{\partial u_r}{\partial z} +$$

$$c_{33}\frac{\partial^2 u_z}{\partial z^2} + c_{44}\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2} + c_{55}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) = \rho\frac{\partial^2 u_z}{\partial t^2} . \qquad \text{(c) (2:16) (cont)}$$

Finally, for isotropy, the displacement equations of motion take the compact form

$$\frac{1}{2}(c_{11}-c_{12})u_{i,jj} + \frac{1}{2}(c_{11}+c_{12})u_{j,ji} = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad (i=1,2,3) \qquad (2:17)$$

CHAPTER III

1. On the Use of Displacement Potential Functions to Solve the Equations of Motion

The classical general solution of the isotropic equations of motion was established by Poisson [50] (1829). He showed that every (sufficiently regular) solution of

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} = \frac{\rho}{\mu} \ddot{u}_i \tag{3:1}$$

admits the representation

$$u_i(x,t) = u_i'(x,t) + u_i''(x,t), \tag{a}$$

$$u_i' = \varphi_{,i} \quad , \quad u_{i,i}'' = 0, \tag{b}$$

(3:2)

where $\varphi(x,t)$ and $u_i''(x,t)$ satisfy the wave equations

$$\Box_1^2 \varphi = 0 \quad , \qquad \Box_2^2 u_i'' = 0 \tag{3:3}$$

in which

$$\Box_n^2 = \nabla^2 - \frac{1}{c_n^2} \frac{\partial^2}{\partial t^2}, \quad (n = 1,2), \tag{a}$$

$$c_1^2 = \left(\frac{2-2\nu}{1-2\nu} \right) \frac{\mu}{\rho} \quad , \quad c_2^2 = \frac{\mu}{\rho} \quad , \tag{b}$$

(3:4)

and ∇^2 is the Laplace operator.

Poisson thus established the fundamental fact that a complete solution of (3:1) can be expressed as a superposition of an irrotational and equivoluminal motion.

The first appearance of the displacement field described in terms of potentials, that is, the Stokes-Helmholtz resolution,

$$u_i = \varphi_{,i} + \epsilon_{ijk}\psi_{k,j} , \qquad (3:5)$$

was apparently due to Lame [51] (1852), who noted that every vector of the form (3:5) satisfies (3:1) if

$$\Box_1^2\varphi = 0 , \qquad \Box_2^2\psi_i = 0 \qquad (3:6)$$

Notice that (3:5) and (3:2) differ only in the definition of the rotational portion of the displacement field.

Clebsch [52] (1863) stated, and later Somigliana [53] (1892), Tedone [54] (1897), and Duhem [55] (1898) independantly proved, that any solution of the equations of motion (3:1) admits the representation (3:5) and (3:6). These facts, along with Duhem's version of the completeness theorem, are given in a paper by Sternberg [56]. Duhem's completeness theorem without proof is included here:

Let $u_i(x_1,x_2,x_3,t)$ be a particular solution of

$$\tfrac{1}{2}(c_{11}-c_{12})u_{i,jj} + \tfrac{1}{2}(c_{11}+c_{12})u_{j,ji} = \rho\ddot{u}_i \quad , \quad (i=1,2,3), \qquad (2:17)$$

in a region of space D and for $t_1 < t < t_2$. Then there exists a scalar function $\varphi(x_1,x_2,x_3,t)$ and a vector function $\psi_i(x_1,x_2,x_3,t)$ such that $u_i(x_1,x_2,x_3,t)$ is represented by (3:5), with

$$\psi_{i,i} = 0, \qquad (3:7)$$

and φ and ψ_i satisfy

$$\Box_1^2\varphi = 0 , \qquad \Box_2^2\psi_i = 0 , \qquad (3:3)$$

where

$$\Box_n^2 = \nabla^2 - \frac{1}{c_n^2}\frac{\partial^2}{\partial t^2} , \ (n=1,2), \qquad \text{(a) (3:4)}$$

and

$$C_1^2 = \left(\frac{2-2\nu}{1-2\nu}\right)\frac{\mu}{\rho}, \qquad C_2^2 = \frac{\mu}{\rho} \qquad \text{(b) (3:4) (cont)}$$

The operator in (3:4) is referred to as the wave operator, and the velocities C_1 and C_2 in (3:4) are, respectively, the dilatational and equivoluminal wave speeds in an infinite, linearly elastic, homogeneous isotropic solid.

Equations (3:5) and (3:7), taken together, indicate that u_i may be represented in terms of three independent potential functions: φ and two of the ψ_i. It is to be noted that u_i given by (3:5) satisfies (3:1) provided (3:3) hold, even if ψ_i is not solenoidal. Consequently, equation (3:7) may be relaxed to the form

$$\psi_{i,i} = F(x_1, x_2, x_3, t). \qquad (3:8)$$

This representation has been used in many of the contemporary investigations, for example, [21,28,29,30], etc., in establishing the complete solution of (3:1) when u_i represent the displacement field for infinite trains of harmonic waves.

Shifting attention now from isotropy to anisotropy, it is early noted that the wealth of formal solutions, so prevalent in the isotropic theory, is almost entirely lacking. Apparently only the case of transverse isotropy has successfully been treated by the use of potentials. Lekhnitsky [57] and Elliot [58] solved the displacement equations of torsionless, axisymmetric equilibrium for several important problems in terms of a single stress function satisfying a fourth-order partial differential equation. Eubanks and Sternberg [59] developed a completeness proof regarding the use of this

potential solution . Mirsky [38] subsequently gave a completeness theorem extending the applicability of the Lekhnitsky potential solution to include the nonsymmetric elastostatic problem.

Mirsky [38] also provided a general theorem concerning the solution of the elastodynamic problem in transversely isotropic media. Through use of a set of potential functions defining the displacement field according to the following equations

$$u_r = \frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,,$$

$$u_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{\partial V}{\partial r}\,, \qquad (3:9)$$

and

$$u_z = \frac{\partial W}{\partial z}\,,$$

he showed that the equations of motion for a transversely isotropic solid are satisfied if, in turn, U, V, and W satisfy the differential equations

$$\nabla_1^2\left(c_{66}\nabla_1^2 V + c_{44}\frac{\partial^2 V}{\partial z^2} - \rho\frac{\partial^2 V}{\partial t^2}\right) = 0 \qquad \text{(a)}$$

and (3:10)

$$\Lambda U = \Lambda W = 0\,. \qquad \text{(b)}$$

Here

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\,, \qquad \text{(a)}$$

and (3:11)

$$\Lambda = \nabla_1^4 + \left[(c_{11}c_{33} - c_{13}^2 - 2c_{44}c_{13})/c_{11}c_{44}\right]\frac{\partial^2}{\partial z^2}\nabla_1^2 - \left[\rho(c_{11}+c_{44})/c_{11}c_{44}\right]\frac{\partial^2}{\partial t^2}\nabla_1^2 + \frac{c_{33}}{c_{11}}\frac{\partial^4}{\partial z^4} - (\text{cont}) \qquad \text{(b)}$$

(cont) $-\left[\rho(c_{33}+c_{44})/c_{11}c_{44}\right]\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial t^2}+\frac{\rho^2}{c_{11}c_{44}}\frac{\partial^4}{\partial t^4}$ (b) (3:11) (cont)

As will subsequently be shown in section 2 of this chapter, the potentials U, V, and W used above by Mirsky follow closely from the Stokes-Helmholtz resolution of the vector displacement field. However, the third of (3:9) represents a minor (and nonsimplifying) departure from the usual specification of the Helmholtz potentials.

Any attempt to factor the operator Λ of equation (3:11) into a product of two second order operators leads to a system of algebraic constraints between the elastic constants, which apparently are only satisfied by the condition of isotropy. Hence, any solution u_i of (2:15), in a region of space D and for $t_1 < t < t_2$, may be represented by (3:9) provided V satisfies (3:10a) and U and W satisfy (3:10b). It is readily apparent that the wave operators no longer play a rôle in the general solutions.

Equations (3:11) are indicative of the mathematical formidity of the elastodynamic problem in anisotropic linear elasticity. No formally acknowledged potential solution (Galerkin, Iekhnitsky, etc.) has been successfully applied to even so simple an anisotropy as that under discussion in this paper. By successful application is meant, of course, a rendering of the elastodynamic problem into a solvable system of equations, capable of being shown to provide the complete, general solution of the problem.

In applying the Stokes-Helmholtz resolution of the vector

displacement field to the isotropic equations of motion for an infinite body, it is natural to obtain, in uncoupled form, the effects of dilatational and rotational motion. In an infinite isotropic medium the displacement vectors associated with dilatational and rotational waves are colinear with, or orthogonal to, the wave front vector, respectively. However, in a general anisotropic infinite continuum, the three displacement vectors associated with the equivoluminal and dilatational waves have no such orientation and the displacement vector, which is colinear with the wave front vector, is generally comprised of both equivoluminal and dilatational displacement components [60]. Furthermore, the two equivoluminal wave velocities are generally dissimilar in an anisotropic body. The magnitudes of the three wave velocities, in addition, generally depend on the direction of propagation within the continuum. These facts imply that in a general anisotropic continuum the equivoluminal and dilatational components of motion will be coupled.

When solutions of the equations of motion are sought in cylindrical coordinates for a transversely isotropic material, the equations are as complex as those for an orthotropic material, if the elastic symmetry axis is not chosen to coincide with the z axis of cylindrical coordinates. No substitution, in terms of potential functions, has been found for the displacement components which renders the orthotropic equations soluble. In fact, use of the conventional resolutions, for example (3:9), for the displacements results in a system

of three third order partial differential equations with variable coefficients, each involving the three potentials. Due to the variable coefficients in the equations it is not possible to organize the three equations into useful operator formulation, since the operators do not commute. Furthermore, among the three resulting equations there is no degree of similarity to aid in the solution. Consequently, though it is conceptually possible to manipulate the system of equations to obtain a new set involving each potential function separately, it is no longer possible to expect each potential to satisfy the same higher order differential equation as is the case if the differential operators commute [61,62].

Frequently simplifications in the governing differential equations can be obtained by assuming the functional form in which certain variables enter the formulation. Additionally, when potential function formulations do not aid the solution process, it is customary to seek particular solutions of the equations of motion in terms of the displacements directly. However, except for the simplest of anisotropies, these two reductions in complexity are of little aid. This is easily illustrated by the following (classical) particular solution.

$$\begin{aligned} u_r &= U(r)\cos n\theta \cos(kz - \omega t), \\ u_\theta &= V(r)\sin n\theta \cos(kz - \omega t), \\ u_z &= W(r)\cos n\theta \sin(kz - \omega t), \end{aligned} \tag{3:12}$$

when introduced in equations (2:16). Here $k = 2\pi/\bar{\lambda}$ is the wave number and $\bar{\lambda}$ the wavelength; ω is the natural frequency. The phase velocity of the wave is given by $v = \omega/k$.

Equation (3:12) reduce (2:16) to

$$c_{11}\frac{d^2U}{dr^2} + \frac{c_{11}}{r}\frac{dU}{dr} + (\rho\omega^2 - c_{55}k^2)U - (c_{66}n^2 + c_{22})\frac{1}{r^2}U +$$

$$+(c_{12}+c_{66})\frac{n}{r}\frac{dV}{dr} - (c_{22}+c_{66})\frac{n}{r^2}V + (c_{13}+c_{55})k\frac{dW}{dr} + (c_{13}-c_{23})\frac{k}{r}W = 0, \quad (a)$$

$$c_{66}\frac{d^2V}{dr^2} + \frac{c_{66}}{r}\frac{dV}{dr} + (\rho\omega^2 - c_{44}k^2)V - (c_{22}n^2 + c_{66})\frac{1}{r^2}V - \quad (3:13)$$

$$-(c_{12}+c_{66})\frac{n}{r}\frac{dU}{dr} - (c_{22}+c_{66})\frac{n}{r^2}U - (c_{23}+c_{44})n\frac{k}{r}W = 0, \quad (b)$$

$$c_{55}\frac{d^2W}{dr^2} + c_{55}\frac{dW}{dr} + (\rho\omega^2 - c_{33}k^2)W - c_{44}\frac{n^2}{r^2}W -$$

$$-(c_{13}+c_{55})k\frac{dU}{dr} - (c_{23}+c_{55})\frac{k}{r}U - (c_{23}+c_{44})\frac{nk}{r}V = 0 \quad (c)$$

The order of differentiation of the above equations is one less than in those obtained when the displacements are represented by potentials. In spite of this fact, the system (3:13) involves essentially all the complications previously discussed. Both Mirsky [34,35] and Nowinski [37] attacked the axisymmetric (n = 0) form of these equations with the Frobenius power series method.

2. A Completeness and Representation Theorem for General Harmonic Waves in Transversely Isotropic Bodies

Attention will be focused now on the equations of motion, that is, equations (2:15), for waves traveling in the direction of the axis of symmetry in transversely isotropic materials. For this condition a completeness theorem and a representation for the displacement field will be developed in detail.

Before proceeding with the theorem, the following specialization of Almansi's [63] theorems on biharmonic functions

is stated. If

$$(\nabla_1^2 + p^2)(\nabla_1^2 + q^2)g(r,\theta) = 0$$

in some (not necessarily simply connected) region D, then $g(r,\theta)$ can be represented as

$$g(r,\theta) = g_1(r,\theta) + g_2(r,\theta), \qquad (a)$$

where (3:14)

$$(\nabla_1^2 + p^2)g_1 = 0, \quad (\nabla_1^2 + q^2)g_2 = 0, \qquad (b)$$

and ∇_1^2 is given in (3:11a).

Theorem: Let u_i (i = r,θ,z) be a solution of the equations of motion (2:15) of a transversely isotropic elastic continuum in some region (not necessarily simply connected) of space D and for $t_1 < t < t_2$. Then, there exist scalar functions $\varphi(r,\theta)\exp(ikz - i\omega t)$ and $f(r,\theta)\exp(ikz - i\omega t)$ such that the displacement field is represented as

$$u_r = \left(\alpha \frac{\partial \varphi(r,\theta)}{\partial r} + \frac{1}{r}\frac{\partial f(r,\theta)}{\partial \theta}\right)\exp i(kz - \omega t),$$

$$u_\theta = \left(\alpha \frac{1}{r}\frac{\partial \varphi(r,\theta)}{\partial \theta} - \frac{\partial f}{\partial r}(r,\theta)\right)\exp i(kz - \omega t), \qquad (3:15)$$

$$u_z = \left(\beta \nabla_1^2 \varphi(r,\theta) + \gamma \varphi(r,\theta)\right)\exp i(kz - \omega t),$$

where α,β,γ are constants with respect to the variables r,θ, and z, ∇_1^2 is given by (3:11a), and φ and f satisfy, respectively,

$$\Box_1^2 \Box_2^2 \varphi = 0, \quad \Box_3^2 f = 0. \qquad (3:16)$$

The wave operators, $\Box_i^2 (i = 1,2,3)$, are defined as

$$\Box_1^2 = \nabla_1^2 + p^2, \quad \Box_2^2 = \nabla_1^2 + q^2, \quad \Box_3^2 = \nabla_1^2 + s^2, \qquad (a)\ (3:17)$$

where

$$\left.\begin{matrix} p^2 \\ q^2 \end{matrix}\right\} = \frac{1}{2c_{11}c_{44}}\left\{ c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2 \pm \right. \qquad \text{(b)(3:17)}$$

$$\left. \pm\sqrt{\left[c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2\right]^2 - 4c_{11}c_{44}(\rho\omega^2 - c_{33}k^2)(\rho\omega^2 - c_{44}k^2)} \right\},$$

in which the "+" and "−" signs are associated with p^2 and q^2, respectively, and

$$s^2 = \frac{1}{c_{44}}\left(\rho\omega^2 - c_{44}k^2\right). \qquad \text{(c) (3:17)}$$

Proof: According to the Stokes-Helmholtz theorem [64,65], every sufficiently smooth vector field u_i may be written as

$$u_i = U_{,i} + \epsilon_{ijk}\tilde{V}_{k,j} \quad , \quad \tilde{V}_{k,k} = 0 \qquad (3:18)$$

Expanding the vector (3:18) into components yields

$$u_r = \left(\frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial V_z}{\partial\theta} - ikV_\theta\right)\exp i(kz - \omega t),$$

$$u_\theta = \left(\frac{1}{r}\frac{\partial U}{\partial\theta} - \frac{\partial V_z}{\partial r} + ikV_r\right)\exp i(kz - \omega t), \qquad \text{(a)}$$

$$u_z = \left(ikU + \frac{\partial V_\theta}{\partial r} + \frac{1}{r}V_\theta - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\exp i(kz - \omega t), \qquad (3:19)$$

and

$$V_{k,k} = \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r}V_r = 0. \qquad \text{(b)}$$

In equation (3:19b) the term $\partial V_z/\partial z$ has been eliminated by taking into account the fact that the vector $\tilde{V}_i$, satisfying the first of (3:18), may contain an arbitrary gradient quantity. That is

$$\tilde{V}_i = V_i + \tilde{q}_{,i},$$

$$\tilde{V}_{i,i} = V_{i,i} + \tilde{q}_{,ii} = \left(\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z}\right) + \tilde{q}_{,ii},$$

where $q_{,ii}$ is chosen equal to $-\partial V_z/\partial z$. Consequently, the

divergence of $\tilde{V}_i$ vanishes if (3:19b) is satisfied. This may be accomplished by choosing

$$V_r = \frac{1}{r}\frac{\partial \tilde{Z}}{\partial \theta} \quad , \quad V_\theta = -\frac{\partial \tilde{Z}}{\partial r}$$

Using this substitution, and subsequently letting

$$\begin{aligned} F(r,\theta) &= U(r,\theta) + ik\tilde{Z}(r,\theta), \\ G(r,\theta) &= V_\theta(r,\theta), \end{aligned} \tag{3:20}$$

equations (3:19) reduce to

$$\begin{aligned} u_r &= \left(\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial \theta}\right)\exp i(kz-\omega t), \\ u_\theta &= \left(\frac{1}{r}\frac{\partial F}{\partial \theta} - \frac{\partial G}{\partial r}\right)\exp i(kz-\omega t), \\ u_z &= \left(ikF + \lambda^2\tilde{Z}\right)\exp i(kz-\omega t), \end{aligned} \tag{3:21}$$

with $\lambda^2 = k^2 - \nabla_1^2$.

Using equations (3:21) in (2:15) yields

$$\frac{\partial}{\partial r}\left[c_{11}\nabla_1^2 F + (\rho\omega^2 - c_{44}k^2)F - k^2(c_{13}+c_{44})\left(F + \frac{\lambda^2}{ik}\tilde{Z}\right)\right] +$$
$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left[c_{66}\nabla_1^2 G + (\rho\omega^2 - c_{44}k^2)G\right] = 0, \qquad (a)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left[c_{11}\nabla_1^2 F + (\rho\omega^2 - c_{44}k^2)F - k^2(c_{13}+c_{44})\left(F + \frac{\lambda^2}{ik}\tilde{Z}\right)\right] -$$
$$- \frac{\partial}{\partial r}\left[c_{66}\nabla_1^2 G + (\rho\omega^2 - c_{44}k^2)G\right] = 0, \qquad (b) \tag{3:22}$$

and

$$\left[(c_{13}+c_{44})\nabla_1^2 F + \left\{c_{44}\nabla_1^2 + (\rho\omega^2 - c_{33}k^2)\right\}\left(F + \frac{\lambda^2}{ik}\tilde{Z}\right)\right] = 0. \quad (c)$$

Appropriate differentiation of the first two of (3:22) gives

$$\nabla_1^2\left[c_{11}\nabla_1^2 F + (\rho\omega^2 - c_{44}k^2)F - k^2(c_{13}+c_{44})\left(F + \frac{\lambda^2}{ik}\tilde{Z}\right)\right] = 0,$$

$$\nabla_1^2\left[c_{66}\nabla_1^2 G + (\rho\omega^2 - c_{44}k^2)G\right] = 0,$$

or

$$\left[c_{11}\nabla_1^2 F + (\rho\omega^2 - c_{44}k^2)F - k^2(c_{13}+c_{44})\left(F + \frac{\lambda^2}{ik}\tilde{Z}\right)\right] = h_1(r,\theta), \quad (a)$$

and

$$\left[c_{66}\nabla_1^2 G + (\rho\omega^2 - c_{44}k^2)G\right] = h_2(r,\theta), \quad (b) \quad (3:23)$$

with

$$\nabla_1^2 h_1(r,\theta) = \nabla_1^2 h_2(r,\theta) = 0. \quad (c)$$

Substitution of (3:23a and b) in (3:22) relates $h_1(r,\theta)$ and $h_2(r,\theta)$ such that

$$\frac{\partial h_1}{\partial r} = -\frac{1}{r}\frac{\partial h_2}{\partial \theta}, \quad \frac{1}{r}\frac{\partial h_1}{\partial \theta} = \frac{\partial h_2}{\partial r}. \quad (3:24)$$

Equations (3:22c) and (3:23a) form a set of relations coupling F and $\tilde{Z}$:

$$L_1 F - L_2 \frac{\lambda^2}{ik}\tilde{Z} = h_1(r,\theta), \quad (a)$$

and

$$L_3 F + L_4 \frac{\lambda^2}{ik}\tilde{Z} = 0, \quad (3:25)$$

where

$$L_1 = c_{11}\nabla_1^2 + (\rho\omega^2 - (c_{13}+2c_{44})k^2),$$

$$L_2 = k^2(c_{13}+c_{44}), \quad (b)$$

$$L_3 = (c_{13}+2c_{44})\nabla_1^2 + (\rho\omega^2 - c_{33}k^2),$$

$$L_4 = c_{44}\nabla_1^2 + (\rho\omega^2 - c_{33}k^2)$$

Consequently,

$$(L_1L_4 + L_2L_3)F = L_4h_1 = (\rho\omega^2 - c_{33}k^2)h_1 \qquad (a)$$

and (3:26)

$$\frac{\lambda^2}{ik}(L_1L_4 + L_2L_3)\tilde{Z} = -L_3h_1 = -(\rho\omega^2 - c_{33}k^2)h_1\,; \qquad (b)$$

and

$$L_1L_4 + L_2L_3 = c_{11}c_{44}\nabla_1^2\nabla_1^2 +$$

$$+\left[c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2\right]\nabla_1^2 + \qquad (3:27)$$

$$+(\rho\omega^2 - c_{33}k^2)(\rho\omega^2 - c_{44}k^2).$$

Solutions of (3:26) are

$$F = F^{(1)} + F^{(2)}\,,$$
$$\tilde{Z} = \tilde{Z}^{(1)} + \tilde{Z}^{(2)};\ Z = \lambda^2\tilde{Z}/ik\,, \qquad (3:28)$$

where

$$\left(L_1L_4 + L_2L_3\right)\left\{F^{(1)}, \tilde{Z}^{(1)}\right\} = 0 \qquad (3:29)$$

and

$$F^{(2)} = h_1(r,\theta)/\rho\omega^2 - c_{44}k^2\,,$$
$$\tilde{Z}^{(2)} = -h_1(r,\theta)/\rho\omega^2 - c_{44}k^2 \qquad (3:30)$$

In view of (3:27), equation (3:29) may be written as a factored operator if

$$(\nabla_1^2 + p^2)(\nabla_1^2 + q^2) = \frac{1}{c_{11}c_{44}}(L_1L_4 + L_2L_3),$$

or

$$c_{11}c_{44}(p^2+q^2) = c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2,$$

and

$$c_{11}c_{44}p^2q^2 = (\rho\omega^2 - c_{33}k^2)(\rho\omega^2 - c_{44}k^2).$$

Solving the above for p^2 and q^2 yields:

$$\begin{Bmatrix} p^2 \\ q^2 \end{Bmatrix} = \frac{1}{2c_{11}c_{44}} \left\{ c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2 \pm \right. \tag{3:31}$$
$$\left. \pm \sqrt{\left[c_{11}(\rho\omega^2 - c_{33}k^2) + c_{44}(\rho\omega^2 - c_{44}k^2) + k^2(c_{13}+c_{44})^2\right]^2 - 4c_{11}c_{44}(\rho\omega^2 - c_{33}k^2)(\rho\omega^2 - c_{44}k^2)} \right\}.$$

Accordingly, the general solutione for $F^{(1)}$ and $Z^{(1)}$ may be represented as

$$F^{(1)} = F_1 + F_2 , \tag{a}$$
$$Z^{(1)} = Z_1 + Z_2 , \tag{b} \tag{3:32}$$

with

$$(\nabla_1^2 + p^2)\{F_1, Z_1\} = 0 ; \ (\nabla_1^2 + q^2)\{F_2, Z_2\} = 0 \tag{c}$$

Thus

$$F = F_1 + F_2 + F^{(2)}, \tag{a}$$
$$Z = Z_1 + Z_2 + Z^{(2)}. \tag{b} \tag{3:33}$$

Equation (3:33a) with (3:32c) is used in (3:23a) to find a repreeentation for Z. This yields

$$-(c_{13}+c_{44})k^2 Z = h_1(r,\theta) + c_{11}(p^2 F_1 + q^2 F_2) - (\rho\omega^2 - (c_{13}+2c_{44})k^2)(F_1 + F_2 + F^{(2)}) \tag{3:34}$$

Using this expreesion for Z, and (3:33a) for F, in (3:22c) gives:

$$0 = -(c_{13}+2c_{44})(p^2 F_1 + q^2 F_2) + (\rho\omega^2 - c_{33}k^2)(F_1 + F_2 + F^{(2)}) -$$

$$-\frac{c_{44}}{k^2(c_{13}+c_{44})}\left[-c_{11}(p^2 F_1 + q^2 F_2) + (\rho\omega^2 - (c_{13}+2c_{44})k^2)(p^2 F_1 + q^2 F_2)\right] -$$

$$-\frac{(\rho\omega^2 - c_{33}k^2)}{k^2(c_{13}+c_{44})}\left[h_1(r,\theta) + c_{11}(p^2 F_1 + q^2 F_2) - (\rho\omega^2 - (c_{13}+2c_{44})k^2)(F_1 + F_2 + F^{(2)})\right],$$

which, using (3:31), vanishes identically.

Turning now to (3:23b), a solution is given by

$$G = G_1 + G^{(1)}, \tag{a} \tag{3:35}$$

where

$$\nabla_1^2 G_1 + \frac{\rho\omega^2 - c_{44}k^2}{c_{66}} G_1 = (\nabla_1^2 + \beta^2) G_1 = 0, \quad G^{(1)} = \frac{h_2(r,\theta)}{\rho\omega^2 - c_{44}k^2}. \quad \text{(b)} \qquad \begin{matrix}(3:35)\\(\text{cont})\end{matrix}$$

A representation for the displacement field is obtained using (3:33a), (3:34), and (3:35) in (3:21) such that

$$u_r = \left(\frac{\partial F}{\partial r} + \frac{\partial F_2}{\partial r} + \frac{\partial F^{(3)}}{\partial r} + \frac{1}{r}\frac{\partial G_1}{\partial \theta} + \frac{1}{r}\frac{\partial G^{(1)}}{\partial \theta}\right) \exp i(kz - \omega t),$$

$$u_\theta = \left(\frac{1}{r}\frac{\partial F}{\partial \theta} + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{1}{r}\frac{\partial F^{(3)}}{\partial \theta} - \frac{\partial G_1}{\partial r} - \frac{\partial G^{(1)}}{\partial r}\right) \exp i(kz - \omega t),$$

$$u_z = \left(ik(F + F_2 + F^{(3)}) + \frac{ik}{\rho(c_{13}+c_{44})}\left[-h_1(r,\theta) - c_{11}(\rho^2 F + q^2 F_2) + \right.\right.$$
$$\left.\left. + (\rho\omega^2 - c_{44}k^2)(F + F_2 + F^{(3)}) - (c_{13}+c_{44})k^2(F + F_2 + F^{(3)})\right]\right) \exp i(kz - \omega t).$$

Using equations (3:24), (3:30), and the second of (3:35b), the following simplifications are made:

$$\frac{1}{r}\frac{\partial G^{(1)}}{\partial \theta} + \frac{\partial F^{(3)}}{\partial r} = 0,$$

$$-\frac{\partial G^{(1)}}{\partial r} + \frac{1}{r}\frac{\partial F^{(3)}}{\partial \theta} = 0,$$

and

$$u_z = \frac{1}{k}\left(\frac{-c_{11}}{c_{13}+c_{44}}(\rho^2 F + q^2 F_2) + \frac{\rho\omega^2 - c_{44}k^2}{c_{13}+c_{44}}(F + F_2)\right) \exp i(kz - \omega t).$$

Define

$$q(r,\theta) = \frac{F + F_2}{c_{13}+c_{44}} = \frac{k}{c_{13}+c_{44}}\varphi_1 + \varphi_2, \qquad \text{(a)}$$
$$\qquad\qquad (3:36)$$
$$f(r,\theta) = G_1(r,\theta), \qquad \text{(b)}$$

where

$$\square_1^2 \square_2^2 \varphi = 0, \qquad \square_3^2 f = 0, \qquad (3:37)$$

and

$$\square_i^2 = \nabla_i^2 + \beta_i^2, \qquad (3:38)$$

$$\Box_2^2 = \nabla_1^2 + q^2 \,, \qquad \Box_3^2 = \nabla_1^2 + s^2 \,, \qquad \text{(3:38) (cont)}$$

with p^2 and q^2 as given by (3:31), and s^2 by (3:35).

Then

$$u_r = \left((c_{13}+c_{44})\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\right)\exp i(kz-\omega t),$$

$$u_\theta = \left((c_{13}+c_{44})\frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial f}{\partial r}\right)\exp i(kz-\omega t), \qquad \text{(3:39)}$$

and

$$u_z = \frac{1}{k}\left(c_n\nabla_1^2\varphi + c_{44}\left(\frac{\rho\omega^2}{c_{44}} - k^2\right)\varphi\right)\exp i(kz-\omega t). \qquad \|$$

3. Alteration of the Representation for Two Special Cases

There are two specific situations that need to be mentioned which might be thought to effect the generality of the above representation: (1) isotropy, and (2) $c_{13}+c_{44}=0$.

If the material body D is isotropic, then $c_{13}\to c_{12}$, $c_{33}\to c_{11}$, $c_{44}\to c_{66}$ and

$$p^2 = \frac{\rho\omega^2}{c_n} - k^2 \,, \quad q^2 = \frac{\rho\omega^2}{c_n} - k^2 \,, \quad s^2 = \frac{\rho\omega^2}{c_{44}} - k^2 \,. \qquad \text{(3:40)}$$

No essential difference exists between this condition and the theorem conditions, hence

$$u_r = \left(\frac{c_{11}+c_{12}}{2}\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\right)\exp i(kz-\omega t),$$

$$u_\theta = \left(\frac{c_{11}+c_{12}}{2}\frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial f}{\partial r}\right)\exp i(kz-\omega t), \qquad \text{(3:41)}$$

$$u_z = \frac{1}{k}\left(c_{11}\nabla_1^2\varphi + \frac{c_{11}-c_{12}}{2}(k_s^2-k^2)\varphi\right)\exp i(kz-\omega t),$$

where

$$k_s^2 = \frac{2\rho\omega^2}{c_{11}-c_{12}} \,, \qquad \text{(a) (3:42)}$$

with

$$\Box_1^2 \varphi_1 = \Box_2^2 \varphi_2 = 0 \qquad \text{(b)}$$

and (3:42) (cont)

$$\Box_1^2 \xi = 0 \qquad \text{(c)}$$

The second case must be allowed since the positive definiteness of the strain energy density function requires only that c_{11}, c_{33}, c_{44}, and $(c_{11} - c_{12})$ be positive. The constants c_{13} and c_{12} can both be negative to extents which are limited by

$$c_{11} - c_{12} > 0 \ , \quad c_{13}^2 - 2c_{13} + c_{12}c_{33} > 0 \qquad (3:43)$$

Consideration of the situation $c_{13} + c_{44} = 0$ is most easily treated by starting with equations (3:22). Thus

$$\frac{\partial}{\partial r}\left[c_{11}\nabla_1^2 + (\rho\omega^2 - c_{44}k^2)\right]F + \frac{1}{r}\frac{\partial}{\partial\theta}\left[c_{66}\nabla_1^2 + (\rho\omega^2 - c_{44}k^2)\right]G = 0,$$

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left[c_{11}\nabla_1^2 + (\rho\omega^2 - c_{44}k^2)\right]F + \frac{\partial}{\partial r}\left[c_{66}\nabla_1^2 + (\rho\omega^2 - c_{44}k^2)\right]G = 0, \qquad (3:44)$$

$$\left[c_{44}\nabla_1^2 + (\rho\omega^2 - c_{33}k^2)\right](F + Z) = 0$$

Equations (3:44) lead to (as done previously)

$$\Box_1^2 F = h_1(r,\theta) \ , \qquad \nabla_1^2 h_1 = 0 \ , \qquad \text{(a)}$$

$$\Box_3^2 G = h_2(r,\theta), \qquad \nabla_1^2 h_2 = 0, \qquad \text{(b)} \qquad (3:45)$$

and

$$\Box_2^2 F + \Box_2^2 Z = 0, \qquad \text{(c)}$$

where the $\Box_i^2 (i = 1,2,3)$ operators are as previously defined with

$$p^2 = \frac{1}{c_{11}}(\rho\omega^2 - c_{44}k^2), \qquad (3:46)$$

$$q_t^2 = \frac{1}{c_{44}}\left(\rho\omega^2 - c_{33}k^2\right) \qquad \begin{matrix}(3:46)\\(\text{cont})\end{matrix}$$

Furthermore, $h_1(r,\theta)$ and $h_2(r,\theta)$ are related by (3:24). Operating on (3:45c) with $\square_1^2$ and noting (3:45a),

$$\square_1^2\square_2^2 Z = -\square_2^2 h_1(r,\theta) = -q_t^2 h_1(r,\theta), \qquad (3:47)$$

since the $\square_i^2 (i = 1,2,3)$ operators commute.

Consequently, Z is represented by

$$Z = Z_1 + Z_2 + Z^{(2)}; \qquad (a)$$

with (3:48)

$$\square_1^2 Z_1 = \square_2^2 Z_2 = 0\,, \quad Z^{(2)} = -\frac{h_1}{p^2} \qquad (b)$$

The solution of (3:45b) is given as

$$G = G_1 + G^{(1)} \qquad (a)$$

with (3:49)

$$\square_3^2 G_1 = 0\,, \quad G^{(1)} = \frac{c_{44}h_1}{\rho\omega^2 - c_{44}k^2} \qquad (b)$$

Equations (3:45a and c) lead directly to the following representation for F in terms of Z and $h_1(r,\theta)$:

$$F = -Z_1 + \frac{h_1}{p^2}\,. \qquad (3:50)$$

Substitution of (3:48a), (3:49a), and (3:50) into (3:21) yields:

$$u_r = \left(-\frac{\partial Z_1}{\partial r} + \frac{1}{r}\frac{\partial G_1}{\partial\theta}\right)\exp i(kz-\omega t),$$

$$u_\theta = \left(-\frac{1}{r}\frac{\partial Z_1}{\partial\theta} - \frac{\partial G_1}{\partial r}\right)\exp i(kz-\omega t), \qquad (3:51)$$

$$u_z = ikZ_1 \exp i(kz-\omega t).$$

On the other hand, if the theorem representation (3:39) is written with account taken of equations (3:36), in the limit as $(c_{13}+c_{44})$ approaches zero, the displacement field takes the form

$$u_r = \left(k\frac{\partial\varphi_1}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\exp i(kz-\omega t), \qquad (a)$$

$$u_\theta = \left(\frac{k}{r}\frac{\partial\varphi_1}{\partial\theta} - \frac{\partial\psi}{\partial r}\right)\exp i(kz-\omega t), \qquad (b) \qquad (3:52)$$

$$u_z = \frac{1}{k}\left\{\lim_{c_{13}+c_{44}\to 0}\left[\frac{c_{44}k}{c_{13}+c_{44}}\nabla_1^2\varphi_1 + \frac{k}{c_{13}+c_{44}}(\rho\omega^2 - c_{44}k^2)\varphi_1\right] + \right.$$
$$\left. + \left[c_{11}\nabla_1^2\varphi_2 + c_{44}\left(\frac{\rho\omega^2}{c_{44}} - k^2\right)\varphi_2\right]\right\}\exp i(kz-\omega t). \qquad (c)$$

It is readily shown that the term involving the limit in (3:52c) vanishes as $(c_{13}+c_{44})$ approaches zero. Although (3:46) gives the proper form for p^2 when $(c_{13}+c_{44})$ is exactly zero, it is useful to show that for small values of $(c_{13}+c_{44})$, p^2 can be written as

$$p^2 = \frac{(\rho\omega^2 - c_{44}k^2)}{c_{11}} + k^2\frac{(c_{13}+c_{44})^2}{2c_{11}c_{44}}\left\{1 + \right.$$
$$\left. + \frac{[2c_{44}(\rho\omega^2/k^2 - c_{33}) + 2c_{44}(\rho\omega^2/k^2 - c_{44}) + (c_{13}+c_{44})^2]}{[c_{44}(\rho\omega^2/k^2 - c_{44}) - c_{11}(\rho\omega^2/k^2 - c_{33})]}\right\} \qquad (3:53)$$

Using equations (3:38) and (3:42b) the term involving the limit in (3:52c) can be written as

$$\lim_{c_{13}+c_{44}\to 0}\frac{[-c_{11}k\rho^2 + k(\rho\omega^2 - c_{44}k^2)]\varphi_1}{c_{13}+c_{44}} .$$

It follows immediately that as $(c_{13}+c_{44})$ vanishes, the coefficient of φ_1 vanishes and u_z may thus be written simply as

$$u_z = -ik\left(\frac{c_{13}+c_{44}}{c_{44}}\frac{\rho\omega^2}{k^2} - \frac{c_{11}c_{33}-c_{13}^2}{c_{44}}\right)\varphi_2\exp i(kz-\omega t) \qquad (3:54)$$

As a consequence of equations (3:52a and b) and (3:54), it is clear that the special case $(c_{13}+c_{44})=0$ is included

in the formulation of the completeness theorem, for if

$$\Psi_1 = -k\varphi_1 \ , \ \Box_1^2 \Psi_1 = \Box_1^2 \varphi_1 = 0 \ ,$$

$$\Psi_2 = -\frac{1}{c_{44}}\left[(c_{11}-c_{44})\frac{\rho\omega^2}{k^2} - c_{11}c_{33} + c_{44}^2\right]\varphi_2 \ , \ \Box_2^2 \Psi_2 = \Box_2^2 \varphi_2 = 0 \ ,$$

and

$$G = f \ , \quad \Box_3^2 G = \Box_3^2 f = 0 \ ,$$

equations (3:51) become just (3:52a and b) and (3:54).

As a consequence of the above results for isotropy and $(c_{13}+c_{44}) = 0$, the representation given in the theorem of the previous section is completely general.

4. Specification of $\varphi(r,\theta)$ and $f(r,\theta)$

Returning to the general representation for the displacement field, (3:39), the scalars φ and f are further restricted to

$$\varphi(r,\theta) = \varphi(r)\left[\cos n\theta + i \sin n\theta\right] ,$$

$$f(r,\theta) = f(r)\left[\sin n\theta - i\cos n\theta\right] . \qquad (3:55)$$

Making the usual specification that the solution be the real part of the above, φ and f are chosen as

$$\varphi(r,\theta,z,t) = \varphi(r)\cos n\theta \cos(kz-\omega t) ,$$

$$f(r,\theta,z,t) = f(r)\sin n\theta \cos(kz-\omega t) , \qquad (a)$$

so that

$$u_r = \left((c_{13}+c_{44})\frac{d\varphi(r)}{dr} + \frac{n}{r}f(r)\right)\cos n\theta \cos(kz-\omega t) ,$$

$$u_\theta = \left(-(c_{13}+c_{44})\frac{n}{r}\varphi(r) - \frac{df(r)}{dr}\right)\sin n\theta \cos(kz-\omega t) , \qquad (b)$$

$$u_z = -\frac{1}{k}\left(c_{11}\nabla_1^2\varphi(r) + c_{44}\left(\frac{\rho\omega^2}{c_{44}} - k^2\right)\varphi(r)\right)\cos n\theta \sin(kz-\omega t) . \qquad (3:56)$$

If, alternatively, the imaginary part of (3:55) is chosen for the solution, then

$$\varphi(r,\theta,z,t) = -\varphi(r)\sin n\theta \sin(kz-\omega t),$$
$$f(r,\theta,z,t) = f(r)\cos n\theta \sin(kz-\omega t), \qquad (a)$$

and

$$u_r = -\left((c_{13}+c_{44})\frac{d\varphi(r)}{dr} + \frac{n}{r}f(r)\right)\sin n\theta \sin(kz-\omega t), \qquad (3:57)$$

$$u_\theta = -\left((c_{13}+c_{44})\frac{n}{r}\varphi(r) + \frac{df(r)}{dr}\right)\cos n\theta \sin(kz-\omega t), \qquad (b)$$

$$u_z = -\frac{1}{k}\left(c_{11}\nabla_1^2\varphi(r) + c_{44}\left(\frac{\rho\omega^2}{c_{44}} - k^2\right)\varphi(r)\right)\sin n\theta \cos(kz-\omega t)$$

Comparison of (3:56) and (3:57) reveals that in the case of axisymmetric displacements ($n = 0$), the torsional displacement field ($u_r = u_z = 0$) is the imaginary part of (3:55) and depends on the single potential f, while the non-torsional field ($u_\theta = 0$) is the real part of the solution and depends only on φ.

Recalling (3:37) and (3:38), in conjunction with (3:56a), it follows that

$$\square_1^2\square_2^2\varphi = \frac{k}{c_{13}+c_{44}}\square_2^2(\square_1^2\varphi_1) + \square_1^2(\square_2^2\varphi_2) = 0,$$

and

$$\square_3^2 f = 0,$$

where now

$$\nabla_1^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \qquad (3:58)$$

Consequently, the operators $\square_1^2 (1 = 1,2,3)$ are now Bessel operators and solutions for φ and f are

$$\varphi(r) = \frac{k}{c_{13}+c_{44}}\left(A_1 J_n(\rho r) + B_1 Y_n(\rho r)\right) + A_2 J_n(qr) + B_2 Y_n(qr), \qquad (a)$$

and (3:59)

$$f(r) = A_3 J_n(sr) + B_3 Y_n(sr) \qquad (b)$$

As a consequence of the completeness theorem given previously, equations (3:55) with (3:59) represent the general solution of the equations of motion (2:15) and (3:56b) and (3:57b) are the resulting displacement fields.

An unusual and significant fact concerning the solution for the equations of motion obtained in this chapter is that the rotational and irrotational portions of the displacement field both rely on φ_1 and φ_2 for definition. In the isotropic theory φ_2 does not appear in the rotational (equivoluminal) field and φ_1 does not appear in the dilatational field. This additional complexity of the anisotropic theory was alluded to in Section 1 of this chapter.

Writing the dilatational and rotational fields as the divergence and curl, respectively, of u_i produces the following expressions

$$u_{i,i} = \left[(c_{13}+c_{44}-c_{11})\nabla_1^2 - \rho\omega^2 + c_{44}k^2\right]\varphi(r)\cos n\theta\cos(kz-\omega t), \qquad (3:60)$$

and

$$\epsilon_{rjk}u_{k,j} = \left(\frac{n}{r}\left[(c_{11}\nabla_1^2 + \rho\omega^2 - c_{44}k^2) - \frac{nk}{r}(c_{13}+c_{44})\right]\varphi - k\frac{df}{dr}\right)\sin n\theta\sin(kz-\omega t), \quad (a)$$

(3:61)

$$\epsilon_{\theta jk}u_{k,j} = \left(\left[\frac{1}{r}k(c_{13}+c_{44}) + \frac{1}{k}(c_{11}\nabla_1^2 + \rho\omega^2 - c_{44}k^2)\right]\frac{d\varphi}{dr} - \frac{nk}{r}f\right)\cos n\theta\sin(kz-\omega t), \quad (b)$$

$$\epsilon_{ijk} u_{k,j} = -\nabla_1^2 f \sin n\theta \cos(kz - \omega t), \qquad \text{(c) (3:61) (cont)}$$

where

$$\varphi = \frac{k}{c_{44} c_{11}} \varphi_1 + \varphi_2$$

Thus, φ_1 and φ_2 are each partially dilatational and equivoluminal in character. Since, in the limit to isotropy, φ_1 passes over to an equivoluminal potential, it will be referred to as quasi-equivoluminal. Similarly, φ_2 will be referred to as quasi-dilatational. Only f(r) retains its singular rôle as an equivoluminal potential in the transition from isotropy to transverse isotropy.

CHAPTER IV

The regular Bessel functions $J_n(pr)$ and $Y_n(pr)$ are oscillatory in nature (for real p) and the number of zeros of J_n in some interval of r, say $r_1 < r < r_2$, is governed by the magnitude of p - in a way similar to that for cos(pr). Hence, the magnitude of p is associated with the number of waves in the interval. The numbers p, q, and s, defined in equations (3:31) and (3:35), will be referred to as radial wave numbers as opposed to the axial wave number k.

The general solution for φ and f (given in equation (3:51)) will be directly affected by changes in the qualitative nature of p, q, and s. For instance, if s is imaginary f(r) ceases to be oscillatory in nature and behaves in an exponential fashion. Similarly, if p and q are complex, then $\varphi(r)$ exhibits an oscillatory, exponentially damped behavior as r is varied. In this chapter an investigation of the range of values to be associated with the radial wave numbers will be conducted. For each range of values found possible, an appropriate form for the solution will be specified.

1. Analysis of the Radial Wave Number s, and f(r)

Since the expression for s^2 is the simplest of the three radial wave numbers, it will be considered first.

$$s^2 = k^2 \frac{c_{55}}{c_{44}} \left(\frac{\rho\omega^2}{c_{44}k^2} - 1 \right) \tag{4:1}$$

If $\rho\omega^2/c_{44}k^2 > 1$, then $s^2 > 0$, otherwise s^2 is zero or negative. The magnitude and sign of s will be indicated by absolute value signs on the parenthesized factor in (4:1) and a sign factor ϵ_3 such that

$$\epsilon_3 = \begin{cases} 1 & \text{if } \rho\omega^2/c_{44}k^2 > 1, \\ -1 & \text{if } \rho\omega^2/c_{44}k^2 < 1. \end{cases} \tag{4:2}$$

Then

$$s^2 = \epsilon_3 k^2 \left| \frac{c_{44}}{c_{55}} \left(\frac{\rho\omega^2}{c_{44}k^2} - 1 \right) \right| = \epsilon_3 k^2 \delta^2,$$

and

$$s = \sqrt{\epsilon_3}\, k\delta. \tag{4:3}$$

In this way the value of δ is always positive and $\sqrt{\epsilon_3}$ is either +1 or i for the conditions listed in (4:2). The value (s) of $\rho\omega^2/k^2$ which causes a radial wave number to vanish is of considerable interest. This number (c_{44} this instance) will be called a turning point number in view of the change in behavior of the Bessel functions when $\rho\omega^2/k^2$ passes from greater than to less than the turning point value.

Using (4:3) and (4:2) in the expression for f(r) yields

$$f(r) = A_3 J_n(sr) + B_3 Y_n(sr).$$

Based on the relations for modified Bessel functions of the first and second kind [66], f(r) assumes the forms

$$f(r) = \begin{cases} A_3 J_n(k\delta r) + B_3 Y_n(k\delta r) & \text{if } \sqrt{\epsilon_3} = 1, \quad \text{(a)} \\ A_3 I_n(k\delta r) + B_3 K_n(k\delta r) & \text{if } \sqrt{\epsilon_3} = i, \quad \text{(b)} \end{cases} \tag{4:4}$$

where the argument of s is either 0 or $\pi/2$. Equations (4:2), (4:3), and (4:4) completely specify the solution for f over all permissable values of real ω and real (positive or negative) k.

The wave equation admits solutions for which the frequency ω may be real, complex, or imaginary. Physically, however, it is only acceptable for the frequency to be real and positive.

Imaginary or complex domains for the axial wave number k are physically accessable and acceptable. If the axial wave number is complex, say $k = a + ib$, the imaginary part of k will enter the wave solutions as an exponential factor causing the amplitudes of the waves to be either increasing or decreasing in magnitude, depending on the sign of b. This attenuation is proportional to the axial coordinate z and, superficially, implies that the sourceless, nondissipative medium (in which the wave is supposed) is either adding to or absorbing the energy transported by the waves. The energy balance, however, is not effected, as Pilant [6] and Oliver [67] explain: when the axial wave number is complex, the traveling waves of given frequency and wavelength occur in pairs, traveling in opposed directions. This phenomena creates a standing disturbance with zero group velocity. Consequently, energy transport, which is proportional to group velocity, is lacking. Standing waves with complex axial wave number occur at the periphery of elastic bodies and have been observed by Oliver [67], Onoe [68], Zemanek

[69], Shaw [70], and others.

When k is a pure imaginary number the propagating component of the wave solution, that is, exp (izRe(k)- iωt), vanishes and is replaced by exp (iωt), which is characteristic of spatially attenuated vibration solutions.

It is emphasized that when k is complex the bearing of the attendant solutions on the case of traveling waves in an infinitely long cylinder is lost since waves traveling in a nondissipative medium must vanish if the amplitude contains an axial attenuation factor of the form exp (-bz).

An oddity in the anisotropic theory, lacking in isotropy, is the occurrence of complex radial wave numbers for real values of ω and k. In this instance, also, there is no energy loss. If p and q are complex for real k, then the wave propagates without axial attenuation. The attenuation is confined to the radial direction of the field.

For completeness the radial wave numbers p, q, and s, and the solutions for q and f will also be presented for the cases k complex or pure imaginary.

Let $k = a + ib$. Then

$$s^2 = \frac{c_{44}}{c_{66}}\left(\rho\frac{\omega^2}{c_{44}} - a^2 + b^2 - i2ab\right). \tag{4:5}$$

If $b \neq 0$, then s^2 is complex and if $k = ib$, s^2 is real and positive for all values of b. Consequently, the representation for f(r) is

$$f(r) = \begin{cases} A_3 J_n(\Im r) + B_3 Y_n(\Im r) & \text{if } a \neq 0, \\ A_3 J_n(s_1 r) + B_3 Y_n(s_1 r) & \text{if } a = 0, \end{cases} \tag{4:6}$$

where

$$\tilde{s} = \sqrt{\frac{c_{44}}{c_{66}}}\sqrt[4]{\left[\frac{\rho\omega^2}{c_{44}} - (\alpha^2 - \beta^2)\right]^2 + 4\alpha^2\beta^2}\ \exp\left\{\frac{1}{2}\tan^{-1}\left(\frac{-2\alpha\beta}{\rho\omega^2/c_{44} - \alpha^2 + \beta^2}\right)\right\}, \qquad \text{(a)}$$

and (4:7)

$$s_t = \sqrt{\frac{c_{44}}{c_{66}}}\sqrt{\frac{\rho\omega^2}{c_{44}} + \beta^2}\ . \qquad \text{(b)}$$

The occurrence of Bessal functions of complex argument in the solution for f(r) may be viewed in the same light as a sinusoidal function of complex argument - for the analytical behavior of the two is roughly comparable. In Section 6 of Chapter V the effects of this type of solution will be fully disclosed. For the moment it is sufficient to indicate that the contribution to the displacement field by such a solution will be of an oscillatory, exponentially decaying nature.

To continue the discussion concerning s and f(r), the representation for f when $\rho\omega^2/k^2 = c_{44}$, that is, the turning point value, will be given. Since $s^2 = 0$ for this value of $\rho\omega^2/k^2$ the Bessal operator reduces to

$$\square_3^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \qquad \text{if } s^2 = 0$$

This is an example of the so called Euler, or equidimensional, differential equation, with solutions

$$f(r) = \begin{cases} A_3 r^n + B_3 r^{-n} & \text{if } n \neq 0, \qquad \text{(a)} \\ A_3 + B_3 \ln r & \text{if } n = 0. \qquad \text{(b)} \end{cases} \qquad (4:8)$$

It is instructive to obtain this representation directly from (4:4). Recognizing that f(r) depends parametrically on the value of k , f(r) may be written as

$$f(r) = A_3(ks)\,J_n(ks\,r) + B_3(ks)\,Y_n(ks\,r)$$

Using the small argument approximations [66] for J_n and Y_n, $f(r)$ becomes

$$f(r) = \begin{cases} A_3(ks)\dfrac{(ks)^n}{2^n n!}r^n - B_3(ks)2^n\dfrac{(n-1)!}{\pi}(ks)^{-n}r^{-n} & \text{if } n \neq 0, \\ A_3(ks) + B_3(ks)\dfrac{2}{\pi}\ln(ks\,r) & \text{if } n = 0 \end{cases} \tag{4:9}$$

Comparing (4:8) and (4:9), the following limits must hold

$$\lim_{ks\to 0} \frac{A_3(ks)(ks)^n}{2^n n!} = A_3(s), \qquad n \geq 0, \tag{a}$$

$$\lim_{ks\to 0} \frac{B_3(ks)2^n(n-1)!}{\pi(ks)^n} = -B_3(s), \qquad n > 0, \tag{b}$$

$$\lim_{ks\to 0} B_3(ks)\frac{2}{\pi}\ln(ks\,r) = B_3(s)\ln r, \qquad n = 0 \tag{c}$$

(4:10)

Notice that (4:10c) is nonlinear in the involvement of the radial coordinate.

The last possibility of interest occurs if k is zero. From (4:1) it follows immediately that

$$s(k=0) = s_c = \sqrt{\epsilon_3}\sqrt{\frac{c_{44}}{c_{44}}}\sqrt{\frac{\rho\omega^2}{c_{44}}}. \tag{4:11}$$

Since s_c is positive without equivocation, ϵ_3 is +1 and equation (4:4a) is the appropriate solution for $f(r)$.

Equations (4:4), (4:6), and (4:8) comprise a complete listing of all possible solutions for $f(r)$ over the widest range of variation of r, k, and ω.

2. Analysis of the Radial Wave Numbers p and q, and $\Phi(r)$

Attention is directed now to equation (3:31) for p^2 and q^2. For convenience, this equation is rewritten here

$$\left.\begin{matrix} p^2 \\ q^2 \end{matrix}\right\} = \frac{k^2}{2c_{11}c_{44}}\left\{c_{11}\left(\rho\frac{\omega^2}{k^2} - c_{33}\right) + c_{44}\left(\rho\frac{\omega^2}{k^2} - c_{44}\right) + (c_{13}+c_{44})^2 \pm \right.$$

$$\left. \pm\left[\left\{c_{11}\left(\rho\frac{\omega^2}{k^2} - c_{33}\right) + c_{44}\left(\rho\frac{\omega^2}{k^2} - c_{44}\right) + (c_{13}+c_{44})^2\right\}^2 - \right. \qquad \text{(a) (4:12)}$$

$$\left. - 4\,c_{11}c_{44}\left(\rho\frac{\omega^2}{k^2} - c_{33}\right)\left(\rho\frac{\omega^2}{k^2} - c_{44}\right)\right]^{1/2}\right\} .$$

If k is real or imaginary p^2 and q^2 are complex (conjugate) if the expression under the radical is negative. If k is complex, then p^2 and q^2 are generally complex. In the case of isotropy the radial wave numbers analogous to p^2 and q^2 can never be complex unless k is complex. This distinction in the range of values for p^2 and q^2 is the most significant difference between transverse isotropy and isotropy, with regard to the radial wave numbers.

In order to expose the different forms of the solution for $\varphi_1(r)$, it is essential to understand the range of values through which p and q may pass, and the order in which these values are passed through. The subsequent discussion is directed to this task. It will frequently be necessary to refer to an interval of values in the range of a variable, say x. If this interval is from a to b, then (a,b), $(a,b]$, and $[a,b]$ will be used to denote $a < x < b$, $a < x \leq b$, and $a \leq x \leq b$, respectively. For convenience, and in congruence with normal physical conditions, it will be assumed henceforth that $c_{33} > c_{44}$. The general stipulation that k is real will be considered initially.

Equation (4:12a) will be written symbolically as

$$\left.\begin{matrix} p^2 \\ q^2 \end{matrix}\right\} = \frac{k^2}{2c_{11}c_{44}}\left(A \pm \sqrt{A^2 - B}\right) \qquad \text{(b) (4:12)}$$

for convenience in the ensuing discussion.

The A term vanishes if

$$\rho\omega^2/k^2\Big|_{L2} = \frac{c_{11}c_{33} + c_{44}^2 - (c_{13}+c_{44})^2}{c_{11}+c_{44}}, \tag{4:13}$$

and the $A^2 - B$ term vanishes if

$$\rho\omega^2/k^2\Big|_{L1},\ \rho\omega^2/k^2\Big|_{L3} = \frac{(c_{11}c_{33} - c_{44}^2)(c_{11}-c_{44}) - (c_{11}+c_{44})(c_{13}+c_{44})^2}{(c_{11}-c_{44})^2} \pm \frac{\pm 2(c_{13}+c_{44})^2\sqrt{c_{11}c_{44}\left[(c_{13}+c_{44})^2 - (c_{11}-c_{44})(c_{33}-c_{44})\right]}}{(c_{11}-c_{44})^2}, \tag{4:14}$$

where the "+" and "-" signs are associated with $\rho\omega^2/k^2\big|_{L1}$ and $\rho\omega^2/k^2\big|_{L3}$, respectively.

Depending on the magnitudes of the elastic constants, the A term of equation (4:12) can pass through zero if $\rho\omega^2/k^2$ is in $[0, c_{33})$. However, A can not be zero with $\rho\omega^2/k^2$ in $[c_{33}, \infty)$ since all three terms in A are strictly positive. By inverting (4:13), the following expression for c_{11} results:

$$c_{11} = \frac{(c_{13}+c_{44})^2 - c_{44}(c_{44} - \rho\omega^2/k^2\big|_{L2})}{c_{33} - \rho\omega^2/k^2\big|_{L2}}. \tag{4:15}$$

A sketch of this equation is given in Figure 1. The figure reveals the following useful inequalities in terms of the elastic constants alone

$$c_{11} \geq \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} \qquad \text{if } \rho\omega^2/k^2\Big|_{L2} \text{ is in } [c_{44}, c_{33}), \tag{a}$$

and (4:16)

$$c_{11} < \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} \qquad \text{if } \rho\omega^2/k^2\Big|_{L2} \text{ is in } [0, c_{44}). \tag{b}$$

The term $A^2 - B$ can never vanish if $\rho\omega^2/k^2$ is in $[c_{33}, \infty)$ as is most easily seen by rewriting $A^2 - B$ as

$$A^2 - B = \left[c_{11}\left(\rho\omega^2/k^2 - c_{33}\right) - c_{44}\left(\rho\omega^2/k^2 - c_{44}\right)\right]^2 + (c_{13}+c_{44})^2\left[(c_{13}+c_{44})^2 + 2c_{11}\left(\rho\omega^2/k^2 - c_{33}\right) + 2c_{44}\left(\rho\omega^2/k^2 - c_{44}\right)\right].$$

Furthermore, when $\rho\omega^2/k^2$ is in $[c_{44}, c_{33})$, $A^2 - B$ again can not vanish. This is seen by examining the form of $A^2 - B$ employed in (4:12e). However, when $\rho\omega^2/k^2$ is in $[0, c_{44})$, $A^2 - B$ may pass through zero (from positive values) and become negative.

The qualitative behavior of equation (4:14) is described by the following observations:

a) If

$$c_{11} > c_{44} + \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}}, \tag{4:17}$$

both $\rho\omega^2/k^2|_{i1}$ $(i = 1,3)$ are complex.

b) If

$$c_{11} = c_{44} + \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}}, \tag{a) (4:18}$$

both $\rho\omega^2/k^2|_{i1}$ and $\rho\omega^2/k^2|_{i3}$ are equal. Their magnitude may be obtained by substituting (4:18a) into (4:14)

$$\rho\frac{\omega^2}{k^2}\Big|_{i1,3} = c_{44}\left(1 - \left[\frac{c_{33}-c_{44}}{c_{13}+c_{44}}\right]^2\right). \tag{b) (4:18}$$

Equation (4:18b) indicates that $(c_{33} - c_{44})^2$ must be smaller than $(c_{13} + c_{44})^2$ if $A^2 - B$ is to vanish for a positive value of $\rho\omega^2/k^2$.

c) If

$$c_{11} < c_{44} + \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}}, \tag{4:19}$$

both $\rho\omega^2/k^2|_{i1}$ $(i = 1,3)$ are real. If the radical term in (4:14) is less than the term

$$\frac{(c_{11}c_{33} - c_{13}^2)(c_{11}-c_{44}) - (c_{11}+c_{44})(c_{13}+c_{44})^2}{(c_{11}-c_{44})^2}, \tag{4:20}$$

and this term is also positive, then both $\rho\omega^2/k^2|_{i1}$ $(i = 1,3)$ are positive. If (4:20) is negative, both $\rho\omega^2/k^2|_{i1}$ $(i = 1,3)$

are negative. Finally, if the radical term of (4:14) is greater than (4:20), then $\rho\omega^2/k^2\big|_{L1}$ is positive and $\rho\omega^2/k^2\big|_{L3}$ is negative.

The quantity $\rho\omega^2/k^2$ can only be a positive, real number because of the stipulation that k be real and the requirement that ω is positive. Applying this criteria, and the qualitative behavior of (4:14) outlined above, the following observations concerning the qualitative behavior of equation (4:12a) hold:

i) If $c_{11} > c_{44} + (c_{13} + c_{44})^2/(c_{33} - c_{44})$, no positive, real value of $\rho\omega^2/k^2$ will cause the radical term of (4:12a) to vanish, and consequently, p^2 and q^2 are real for all ω and k in $[0,\infty)$.

ii) If $c_{11} = c_{44} + (c_{13} + c_{44})^2/(c_{33} - c_{44})$, and if $\rho\omega^2/k^2\big|_{L3}$ is equal or greater than 0, the radical term of (4:12a) will vanish when $\rho\omega^2/k^2 = \rho\omega^2/k^2\big|_{L1,3}$. For this case p^2 and q^2 are always real numbers (positive, or negative) for all ω and k in $[0,\infty)$.

iii) If $c_{11} < c_{44} + (c_{13} + c_{44})^2/(c_{33} - c_{44})$, it is possible for the radical term of (4:12a) to be imaginary for real, positive $\rho\omega^2/k^2$. As outlined in part (c) of the previous paragraph, this condition prevails whenever $\rho\omega^2/k^2\big|_{L1}$ is positive. Moreover, the smallest value of $\rho\omega^2/k^2$ for which the radical term of (4:12a) can remain imaginary is $\rho\omega^2/k^2\big|_{L3}$ or zero, depending on whether $\rho\omega^2/k^2\big|_{L3}$ is positive or negative, respectively. Consequently, p^2 and q^2 are complex (conjugate) numbers when

$$0 \leq \rho\frac{\omega^2}{k^2}\Big|_{L3} < \rho\frac{\omega^2}{k^2} < \rho\frac{\omega^2}{k^2}\Big|_{L1} .$$

The above discussion has been illustrated in Figure 1 by the curves marked (2) and (3). These curves are obtained by inverting (4:14) to form the following inequality involving c_{11}

$$\frac{\left[(c_{13}+c_{44})-\sqrt{c_{44}(c_{44}-\rho\omega^2/k^2)}\right]^2}{c_{33}-\rho\omega^2/k^2} < c_{11} < \frac{\left[(c_{13}+c_{44})+\sqrt{c_{44}(c_{44}-\rho\omega^2/k^2)}\right]^2}{c_{33}-\rho\omega^2/k^2} . \qquad (4:21)$$

Inequality (4:21) gives the restrictions on c_{11}, in terms of elastic constants and $\rho\omega^2/k^2$, such that $A^2 - B$ in (4:12) will be negative. If, in (4:21), $\rho\omega^2/k^2$ is varied from 0 to c_{44} curve (2) in Figure 1 results from the left hand side of (4:21) and (3) from the right hand side. A horizontal line drawn through any value of c_{11} then intersects these two curves in none, one, or two places. These intersections correspond to $\rho\omega^2/k^2\big|_{L3}$ and $\rho\omega^2/k^2\big|_{L1}$, and the larger (or only) value is $\rho\omega^2/k^2\big|_{L1}$. Notice that the horizontal line also intersects curve (1) which is equation (4:15). From this it is to be noted that when $\rho\omega^2/k^2\big|_{L2}$ is in $[0, c_{44})$ it lies intermediate to the values given by (4:14). As previously noted, when c_{11} is greater than $c_{44} + (c_{13} + c_{44})^2/(c_{33} - c_{44})$ there are no intersections with curves (2) and (3), in consonance with the requirement that k be real.

From (4:21), if $\rho\omega^2/k^2 = 0$

$$\frac{c_{13}^2}{c_{33}} < c_{11} < \frac{(c_{13}+2c_{44})^2}{c_{33}}, \qquad (4:22)$$

and if $\rho\omega^2/k^2 = c_{44}$

$$c_{11} = \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} . \qquad (4:23)$$

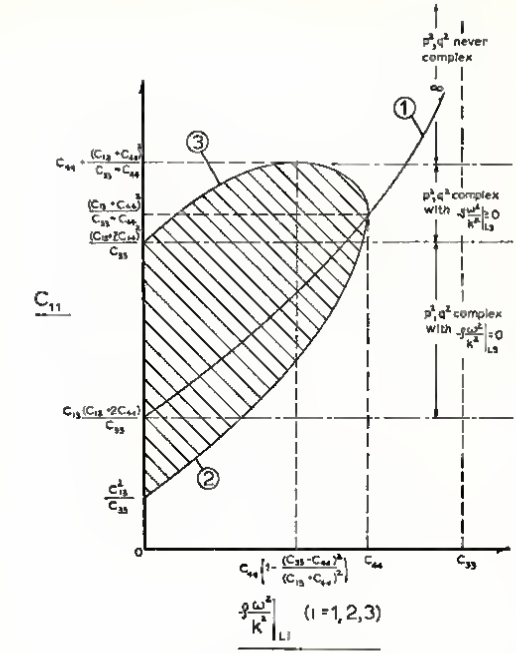

FIGURE 1. c_{11} versus $\rho\omega^2/k^2\Big|_{L1}$ (i = 1,2,3)

Sketch of equation (4:15) (curve ①) and inequality (4:21) (curves ② and ③) showing the limitations on c_{11} for which the radial wave numbers may (the shaded region), or may not be complex.

Equation (4:22) sets limits on c_{11} such that $\rho\omega^2/k^2$ is always negative. Equation (4:23) shows the requirement on c_{11} for which there is no range of values of $\rho\omega^2/k^2$ left-adjacent to c_{44} such that p^2 and q^2 are not complex.

Before returning to a description of the solutions for $\psi(x)$, it is worth while to discuss the ordering of the two ratios

$$\frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} \quad , \quad \frac{(c_{13}+2c_{44})^2}{c_{33}} .$$

Because the elastic constants are independent, and because c_{13} is not limited to positive values, it is not clear under what circumstances one or the other ratio is the larger. By assuming arbitrarily

$$\frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} > \frac{(c_{13}+2c_{44})^2}{c_{33}} , \tag{4:24}$$

and inquiring into the truth of this inequality, the following requirements are found to exist between c_{13}, c_{33}, and c_{44}

$$c_{44}\left[4c_{44}^2+4c_{13}c_{44}-3c_{33}c_{44}+c_{13}^2-2c_{13}c_{33}\right]>0. \tag{4:25}$$

But, the positive definiteness of the strain energy density requires $c_{44}>0$, hence the quadratic must be positive.

With

$$c_{44}^{(1)} = -\frac{1}{8}\left(4c_{13}-3c_{33}\right)-\frac{1}{8}\sqrt{c_{33}\left(8c_{13}+9c_{33}\right)} \qquad \text{(a)}$$

and (4:26)

$$c_{44}^{(2)} = -\frac{1}{8}\left(4c_{13}-3c_{33}\right)+\frac{1}{8}\sqrt{c_{33}\left(8c_{13}+9c_{33}\right)} , \qquad \text{(b)}$$

equation (4:25) yields to the form

$$0<\left(c_{44}-c_{44}^{(1)}\right)\left(c_{44}-c_{44}^{(2)}\right).$$

Thus, if (4:24) holds

$$0 < c_{44} < c_{44}^{(1)} \quad \text{or} \quad c_{44} > c_{44}^{(2)} \qquad \text{if} \quad -9c_{33}/8 \leq c_{13} \leq 0,$$

$$c_{44} > c_{44}^{(2)} \qquad \text{if} \qquad 0 \leq c_{13} \leq 2c_{33}, \tag{4:27}$$

and

$$c_{44} > 0 \qquad \text{if} \qquad 2c_{33} < c_{13}.$$

If (4:24) does not hold, then

$$c_{44}^{(1)} < c_{44} < c_{44}^{(2)} \qquad \text{if} \qquad -9c_{33}/8 \leq c_{13} \leq 0,$$

or (4:28)

$$0 < c_{44} < c_{44}^{(2)} \qquad \text{if} \qquad 0 \leq c_{13} \leq 2c_{33}.$$

When $c_{13} = -9c_{33}/8$ the two branches of the parabola (4:25), given by $c_{44}^{(1)}$ and $c_{44}^{(2)}$, join and (4:24) holds for any c_{33}, c_{44} if $c_{13} < -9c_{33}/8$.

Figure 2 illustrates these facts about (4:24). The coordinate lines have been taken as c_{44}/c_{33} and c_{13}/c_{33}, and since $c_{44} \leq c_{33}$ by assumption, the maximum value of the ordinate can not exceed 1. The requirement, from (4:18), that $\rho\omega^2/k^2|_{L_1 3}$ be positive when c_{11} is given by (4:17) is indicated by the curve labled (1). For material constants with values to the right of this curve, $\rho\omega^2/k^2|_{L_1 3}$ is positive.

Turning now to the radial wave numbers p^2 and q^2, it is noted (see Fig. 1) that only when $\rho\omega^2/k^2|_{L_2}$ is in $[c_{44}, c_{33}]$ is it possible for there not to be a complex range for p^2 and q^2. Transversely isotropic materials for which p^2 and q^2 are always real (for example, magnesium) will be called less anisotropic in analogy with isotropic materials. If $\rho\omega^2/k^2|_{L_2}$ is in $[c_{44}, c_{33}]$ and there are complex zones in the range of p^2 and q^2 for real, positive $\rho\omega^2/k^2$, the anisotropy will be

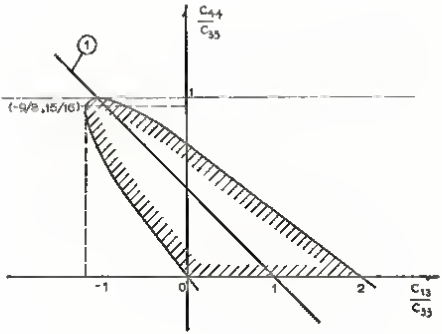

FIGURE 2. Inequality (4:24)

Within the shaded region the reverse of (4:24) holds. Curve (1) delimits the regions of material constants such that to the right of the curve the quantity $\rho\omega^2/k^2|_{1,3}$ of equation (4:16b) is positive.

termed more anisotropic, class I. If, on the other hand, $\rho\omega^2/k^2\big|_{L2}$ is in $[0,c_{44})$, there are always complex zones for p^2 and q^2 and such anisotropies will be referred to as more anisotropic, class II. Class I more anisotropic materials exist when ever the material constants satisfy the following inequality

$$\frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} < c_{11} < c_{44} + \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} \qquad (4:29)$$

Class II anisotropies (always more anisotropic) exist whenever the material constants satisfy

$$c_{44} \leq \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}} \qquad (4:30)$$

On the basis of this description any material may be classified in a manner which brings to mind the behavior of the radial wave numbers, for instance beryllium and zinc are, respectively, class I and class II more anisotropic materials.

The classification has further utility, as will now be discussed. Consider class I materials and recall that the turning point numbers are those values of $\rho\omega^2/k^2$ for which a radial wave number vanishes. Referring to (4:12) it is seen that when $\rho\omega^2/k^2 = c_{33}$, p^2 and q^2 are

$$p^2 = \frac{k^2}{c_{44}c_{33}}\left[c_{44}(c_{33}-c_{44}) + (c_{13}+c_{44})^2\right], \qquad (a)$$

$$q^2 = 0 \qquad (b) \qquad (4:31)$$

Thus $\rho\omega^2/k^2 = c_{33}$ is a turning point number for q.

For $\rho\omega^2/k^2$ in the interval $[c_{44},c_{33})$ the $A^2 - B$ term is always greater than A^2 ($-B$ is positive), hence q^2 is negative throughout the interval, and p^2 remains positive.

Since materials of class I are under consideration, it is apparent that when $\rho\omega^2/k^2 = c_{44}$ the A term is negative and

$$q^2 = -\frac{k^2}{c_{11}c_{44}}\left[c_{44}(c_{33}-c_{44})-(c_{13}+c_{44})^2\right], \qquad \text{(a)}$$

$$p^2 = 0. \qquad \text{(b)} \qquad (4:32)$$

Thus $\rho\omega^2/k^2 = c_{44}$ is a turning point for p.

As $\rho\omega^2/k^2$ decreases into $[0,c_{44})$ both p^2 and q^2 become negative since A is negative and now $A^2 - B$ is less than A^2. If the material is less anisotropic, that is, if

$$c_{11} > c_{44} + \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}},$$

p^2 and q^2 remain negative throughout the interval $[0,c_{44})$; otherwise p^2 and q^2 pass from real numbers to complex conjugate numbers when $\rho\omega^2/k^2 = \rho\omega^2/k^2\big|_{L1}$ and back to negative real numbers when $\rho\omega^2/k^2 = \rho\omega^2/k^2\big|_{L3}$ if $\rho\omega^2/k^2\big|_{L3} > 0$.

For $\rho\omega^2/k^2$ contained in $[\rho\omega^2/k^2\big|_{L1},\infty)$, p and q are real until $\rho\omega^2/k^2$ decreases below their respective turning point numbers, at which point they become imaginary. When $\rho\omega^2/k^2$ is in $(\rho\omega^2/k^2\big|_{L3},\rho\omega^2/k^2\big|_{L1})$ p and q are negative complex conjugate numbers, that is,

$$q = -p^*, \qquad (4:33)$$

where the asterisk means conjugate. The range of the argument of complex q is $\pi/2 < \arg(q) < 3\pi/4$, and for complex p is $\pi/4 < \arg(p) < \pi/2$. Figure 3 illustrates the behavior of p, q, and p^2, q^2 for class I materials when $\rho\omega^2/k^2\big|_{L3} < 0$.

Consider class II materials. As for class I materials, $\rho\omega^2/k^2 = c_{33}$ is a turning point number for q, resulting in equations (4:31) for p^2 and q^2. When $\rho\omega^2/k^2$ is in $[c_{44},c_{33})$

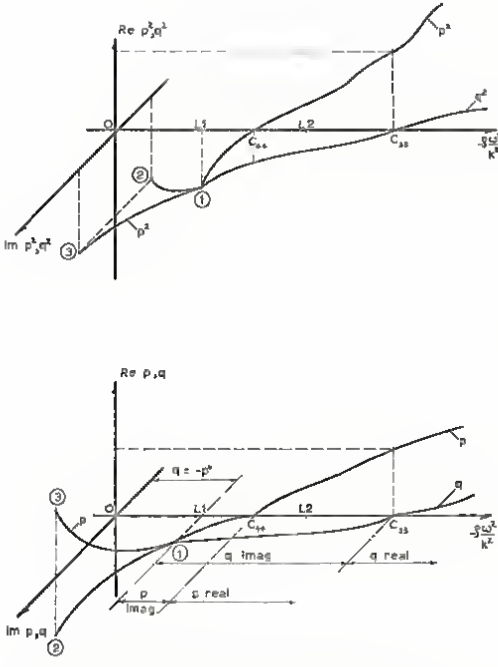

FIGURE 3. Radial Wave Numbers versus $\rho\omega^2/k^2$

The bottom sketch shows the ranges of p and q relative to each other and relative to material constants and $\rho\omega^2/k^2$ for class I more anisotropic materials.

the A term of (4:12) remains positive and $A^2 - \delta$ greater than A^2. Thus, p^2 is positive, q^2 negative. When $\rho\omega^2/k^2 = c_{44}$, q again vanishes and

$$p^2 = \frac{k^2}{c_{11}c_{44}}\left[-c_{11}(c_{33}-c_{44}) + (c_{13}+c_{44})^2\right], \qquad (a)$$

$$q = 0. \qquad (b) \qquad (4:34)$$

As $\rho\omega^2/k^2$ decreases into $[0, c_{44})$, q^2 becomes positive again since now $A^2 - \delta < A^2$ and $A > 0$. Since this is the class II case, there is always a $\rho\omega^2/k^2\big|_{L1}$ such that when $\rho\omega^2/k^2 < \rho\omega^2/k^2\big|_{L1}$, both p^2 and q^2 are complex. When $\rho\omega^2/k^2 = \rho\omega^2/k^2\big|_{L2}$ the real part of p^2 and q^2 vanishes and, as $\rho\omega^2/k^2$ decreases below this value, reappears negatively. As previously discussed, the complex interval extends either to $\rho\omega^2/k^2\big|_{L3}$ or 0. Finally, when $\rho\omega^2/k^2\big|_{L3} \geq \rho\omega^2/k^2 \geq 0$, p^2 and q^2 again become negative real numbers.

For $\rho\omega^2/k^2$ contained in $[\rho\omega^2/k^2\big|_{L1}, \infty)$, p is always real and q passes from real to imaginary to real as $\rho\omega^2/k^2$ passes through the turning point numbers c_{33} and c_{44}. When $\rho\omega^2/k^2$ is in $(\rho\omega^2/k^2\big|_{L3}, \rho\omega^2/k^2\big|_{L1})$, p and q are negative complex conjugates, related as in (4:33). The arguments of p and q are bounded such that $0 < \arg(p) < \pi/2$ and $\pi/2 < \arg(q) < \pi$. Figure 4 shows this behavior for p, q, and p^2, q^2 for class II materials with $\rho\omega^2/k^2\big|_{L3} < 0$. Both Figures 3 and 4 are non-scale sketches, included strictly for illustrative purposes.

Employing the symbology of equation (4:2), with ϵ_1 associated with p^2 and ϵ_2 with q^2, and the extended definitions

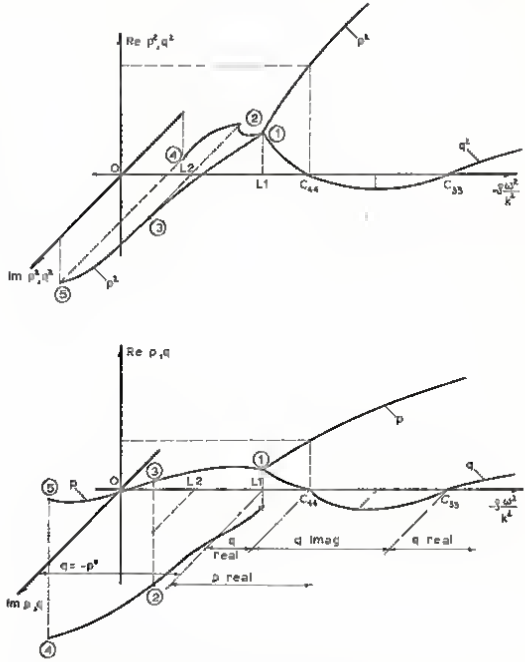

FIGURE 4. Radial Wave Numbers versus $\rho\omega^2/k^2$

The bottom sketch shows the ranges of p and q relative to each other and relative to material constants and $\rho\omega^2/k^2$ for class II more anisotropic materials.

$$\epsilon_1 = \begin{cases} 1 & \text{if } \rho\omega^2/k^2 > c_{44}\,, \\ -1 & \text{if } c_{44} > \rho\omega^2/k^2 \geq \rho\omega^2/k^2\big|_{L1}\,, \\ 1 & \text{if } \rho\omega^2/k^2\big|_{L1} > \rho\omega^2/k^2 > \rho\omega^2/k^2\big|_{L3}\,, \\ -1 & \text{if } \rho\omega^2/k^2\big|_{L3} \geq \rho\omega^2/k^2 \geq 0\,, \end{cases} \quad \text{cl.I} \quad \text{(a)} \tag{4:35}$$

or

$$\epsilon_1 = \begin{cases} 1 & \text{if } \rho\omega^2/k^2 > \rho\omega^2/k^2\big|_{L3}\,, \\ -1 & \text{if } \rho\omega^2/k^2\big|_{L3} \geq \rho\omega^2/k^2 \geq 0\,, \end{cases} \quad \text{cl.II} \quad \text{(b)}$$

and

$$\epsilon_2 = \begin{cases} 1 & \text{if } \rho\omega^2/k^2 > c_{33}\,, \\ -1 & \text{if } c_{33} > \rho\omega^2/k^2 > c_{44}\,, \\ 1 & \text{if } c_{44} > \rho\omega^2/k^2 > \rho\omega^2/k^2\big|_{L3}\,, \\ -1 & \text{if } \rho\omega^2/k^2\big|_{L3} \geq \rho\omega^2/k^2 \geq 0\,, \end{cases} \quad \text{cl.I} \quad \text{(a)}$$

or

$$\epsilon_2 = \begin{cases} 1 & \text{if } \rho\omega^2/k^2 > c_{33}\,, \\ -1 & \text{if } c_{33} > \rho\omega^2/k^2 \geq \rho\omega^2/k^2\big|_{L1}\,, \\ 1 & \text{if } \rho\omega^2/k^2\big|_{L1} > \rho\omega^2/k^2 > \rho\omega^2/k^2\big|_{L3}\,, \\ -1 & \text{if } \rho\omega^2/k^2\big|_{L3} \geq \rho\omega^2/k^2 \geq 0\,, \end{cases} \quad \text{cl.II} \quad \text{(b)} \tag{4:36}$$

the radial wave numbers p and q may be written as

$$p = \sqrt{\epsilon_1}\, k\alpha\,, \quad \text{(a)}$$
$$q = \sqrt{\epsilon_2}\, k\beta\,, \quad \text{(b)} \tag{4:37}$$

where

$$\left.\begin{matrix}\alpha \\ \beta\end{matrix}\right\} = \frac{1}{2c_{44}c_{66}}\left|c_{13}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)+c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)+\left(c_{13}+c_{44}\right)^2 \pm \right. \tag{4:38}$$
$$\left. \pm\sqrt{\left[c_{13}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)+c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)+\left(c_{13}+c_{44}\right)^2\right]^2-4c_{33}c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)}\right|^{1/2},$$

if $\rho\omega^2/k^2$ is contained in $[0, \rho\omega^2/k^2|_{L3}]$ or $[\rho\omega^2/k^2|_{L1}, \infty)$; or

$$\left.\begin{matrix}\alpha \\ \beta\end{matrix}\right\} = B^{1/4}\left(\pm\cos\theta + i\sin\theta\right), \tag{4:39}$$

where

$$\theta = \frac{1}{2}\tan^{-1}\left\{\frac{\sqrt{4c_{11}c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)-\left[c_{11}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)+c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)+(c_{13}+c_{44})^2\right]^2}}{c_{11}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)+c_{44}\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)+(c_{13}+c_{44})^2}\right\}, \tag{4:40}$$

$$B = \left|\frac{1}{c_{11}}\left(\rho\frac{\omega^2}{k^2}-c_{33}\right)\frac{1}{c_{44}}\left(\rho\frac{\omega^2}{k^2}-c_{44}\right)\right|^{1/4},$$

if $\rho\omega^2/k^2$ is contained in $(\rho\omega^2/k^2|_{L3}, \rho\omega^2/k^2|_{L1})$.

Since the regular Bessel functions are analytic functions of real or complex variable z (z = 0 being possible excepted) and real or complex order [66], and since, when p and q are complex the argument of the variable is in the principal range (that is, $|\arg z| < \pi$), the function $\varphi(r)$ may be written as

$$\varphi(r) = \frac{k}{c_{13}+c_{44}}\left(A_1 Z_n(k\alpha r) + B_1 W_n(k\alpha r)\right) + A_2 Z_n(k\beta r) + B_2 W_n(k\beta r). \tag{4:41}$$

Here Z_n and W_n are regular or modified Bessel functions of the first or second kind, respectively, depending on whether ϵ_i (i = 1,2) is +1 or -1. Notice that equation (4:4) may be written in this form, also.

As previously noted in the discussion of f(r), the differential wave operators degenerate when $\rho\omega^2/k^2$ attains the value of a turning point number. Thus $\varphi(r)$ takes the alternate forms

$$\varphi(r) = \begin{cases} k\left(A_1 J_n(k\alpha r) + B_1 Y_n(k\alpha r)\right)/(c_{13}+c_{44}) + A_2 r^n + B_2 r^{-n}, & n>0,\ \beta=0, \\ k\left(A_1 J_0(k\alpha r) + B_1 Y_0(k\alpha r)\right)/(c_{13}+c_{44}) + A_2 + B_2 \ln r, & n=0,\ \beta=0, \end{cases} \tag{a}$$

and (4:42)

$$\varphi(r) = \begin{cases} k\left(A_1 r^n + B_1 r^{-n}\right)/(c_{13}+c_{44}) + A_2 I_n(k\beta r) + B_2 K_n(k\beta r), & n>0,\ \alpha=0, \\ k\left(A_1 + B_1 \ln r\right)/(c_{13}+c_{44}) + A_2 I_0(k\beta r) + B_2 K_0(k\beta r), & n=0,\ \alpha=0, \end{cases} \tag{b}$$

for class I materials, and

$$\varphi(r)=\begin{cases} k\left(A_1 J_n(k\alpha r)+B_1 Y_n(k\alpha r)\right)/(c_{33}+c_{44})+A_2 r^n+B_2 r^{-n}, & n>0;\ \beta=0, \\ k\left(A_1 J_0(k\alpha r)+B_1 Y_0(k\alpha r)\right)/(c_{13}+c_{44})+A_2+B_2 \ln r, & n=0,\ \beta=0, \end{cases} \tag{4:43}$$

for class II materials. Note that (4:43) holds for both turning point values of $\rho\omega^2/k^2$. Equations (4:42) and (4:43) could have been obtained by use of equations (4:10) with ζ replaced by the appropriate argument.

If the materials are more anisotropic the radial wave numbers will be equal when $\rho\omega^2/k^2 = \rho\omega^2/k^2\big|_{L1}$.

$$\alpha^2=\beta^2=\left|\frac{1}{c_{11}}\left(\rho\frac{\omega^2}{k^2}\Big|_{L3}-c_{33}\right)\frac{1}{c_{44}}\left(\rho\frac{\omega^2}{k^2}\Big|_{L1}-c_{44}\right)\right|^{1/2}. \tag{4:44}$$

When (4:44) holds, the differential equation for φ degenerates to

$$\square_1^2\square_1^2\varphi=0\ ,\quad \square_1^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+k^2\alpha^2 \tag{4:45}$$

An appropriate solution [71] of this hyper-Bessel equation is

$$\varphi(r)=A_1 J_n(k\alpha r)+B_1 Y_n(k\alpha r)+r\left(A_2 J_{n+1}(k\alpha r)+B_2 Y_{n+1}(k\alpha r)\right) \tag{4:46}$$

as can be easily verified by substitution into (4:45). Equation (4:46) also holds when

$$\alpha^2=\beta^2=\left|\frac{1}{c_{11}}\left(\rho\frac{\omega^2}{k^2}\Big|_{L3}-c_{33}\right)\frac{1}{c_{44}}\left(\rho\frac{\omega^2}{k^2}\Big|_{L3}-c_{44}\right)\right|^{1/2}, \tag{4:47}$$

but is only applicable if $\rho\omega^2/k^2\big|_{L3}\geq 0$.

If k is set equal to zero both p^2 and q^2 become positive, real functions of ω alone. Because of a requirement for use at a later stage of the analytical development of the problem, the limiting behavior of p^2 and q^2, as $k\to 0$, is presented here as follows. Rearranging terms in (4:12a) the limit, as

$k \to 0$, can be written

$$\underset{k\to 0}{\text{Limit}} \begin{Bmatrix} p^2 \\ q^2 \end{Bmatrix} = \underset{k\to 0}{\text{Limit}} \left\{ \frac{K_1}{2} \rho\frac{\omega^2}{c_{44}} + K_2\frac{k^2}{2} \pm \left[\left(K_1 + \frac{4}{c_{11}}\right)\rho^2\frac{\omega^4}{c_{44}^2} + \right.\right.$$

$$\left.\left. + \left[\left(4\frac{c_{33}}{c_{11}}\left(\frac{c_{44}}{c_{13}}+1\right) + 2K_1K_2\right)\rho\frac{\omega^2}{c_{44}} + \left(K_2^2 - \frac{4c_{33}}{c_{11}}\right)k^2 \right] k^2 \right]^{1/2} \right\},$$

where

$$K_1 = \left(1 + \frac{c_{44}}{c_{11}}\right),\ K_2 = \frac{c_{44}}{c_{11}}\left(\frac{c_{13}}{c_{44}}+1\right)^2 - \frac{c_{44}}{c_{11}} - \frac{c_{33}}{c_{44}}.$$

Employing the first two terms of the binomial series, p^2 and q^2 are given as

$$p^2 \doteq \rho\frac{\omega^2}{c_{44}} + \frac{1}{2}k^2\left\{K_2^2 + \left[\frac{2c_{33}}{c_{11}}\left(\frac{c_{44}}{c_{33}}+1\right) + K_1K_2\right]\Big/\left(1 - \frac{c_{44}}{c_{11}}\right)\right\},$$

$$q^2 \doteq \rho\frac{\omega^2}{c_{11}} + \frac{1}{2}k^2\left\{K_2^2 - \left[\frac{2c_{33}}{c_{11}}\left(\frac{c_{44}}{c_{33}}+1\right) + K_1K_2\right]\Big/\left(1 - \frac{c_{44}}{c_{11}}\right)\right\}$$

It follows, that as $k \to 0$,

$$p^2 \to p_c^2 = \rho\frac{\omega^2}{c_{44}}, \qquad \text{(a)}$$

(4:48)

$$q^2 \to q_c^2 = \rho\frac{\omega^2}{c_{11}}. \qquad \text{(b)}$$

Equation (4:41) is, for this case, the appropriate solution for $\varphi(r)$ with Z_n and W_n regular Bessel functions of the first and second kind, respectively.

Equations (4:41), (4:42), (4:43), and (4:44) describe appropriate solutions for $\varphi(r)$ for all possible values of p and q if k is real.

When k is assumed complex, that is, $k = a + ib$, equation (4:12) assumes the form

$$\begin{Bmatrix} p^2 \\ q^2 \end{Bmatrix} = \frac{1}{c_{11}c_{44}} R \exp i\Theta, \qquad (4:49)$$

where

$$\Theta = -\tan^{-1}\left\{ \frac{abD \mp \frac{\gamma}{2}\sin s}{(a^2-b^2)A \pm \frac{\gamma}{2}\cos s} \right\},$$

$$R = \frac{1}{2}\sqrt{(a^2-b^2)^2A^2 + 4(ab\bar{D})^2 + \bar{\gamma}^2 \pm 2\bar{\gamma}\left[(a^2-b^2)Ac\cos\theta - 2ab\bar{D}\sin\theta\right]},$$

$$\bar{\gamma} = \sqrt[3]{\left[(a^2-b^2)(A^2-B) + 4a^2b^2(4c_{44}c_{33}-\bar{D}^2)\right]^2 + 16(a^2-b^2)^2b^2(E-AD)},$$

$$\theta = \frac{1}{2}\tan^{-1}\left\{\frac{4(a^2-b^2)ab(E-AD)}{(a^2-b^2)^2(A^2-B) + 4a^2b^2(4c_{44}c_{33}-\bar{D}^2)}\right\},$$

$$A = c_{11}\left(\frac{\rho\omega^2}{a^2-b^2} - c_{33}\right) + c_{44}\left(\frac{\rho\omega^2}{a^2-b^2} - c_{44}\right) + (c_{13}+c_{44})^2,$$

$$B = 4c_{11}c_{44}\left(\frac{\rho\omega^2}{a^2-b^2} - c_{33}\right)\left(\frac{\rho\omega^2}{a^2-b^2} - c_{44}\right),$$

$$\bar{D} = (c_{13}+c_{44})^2 - c_{11}c_{33} - c_{44}^2,$$

$$E = 2c_{11}c_{44}\left[c_{44}\left(\frac{\rho\omega^2}{a^2-b^2} - c_{33}\right) + c_{33}\left(\frac{\rho\omega^2}{a^2-b^2} - c_{44}\right)\right]$$

If neither a nor b is zero, p and q are complex. If $b=0$, (4:49) reverts to (4:12), and if $a=0$, (4:49) gives

$$\left.\begin{matrix} p^2 \\ q^2 \end{matrix}\right\} = \frac{k^2}{2c_{11}c_{44}}\left(\underline{A} \mp \sqrt{\underline{A}^2 - \underline{B}}\right), \qquad (4:50)$$

where

$$\underline{A} = c_{11}\left(\rho\frac{\omega^2}{k^2} + c_{33}\right) + c_{44}\left(\rho\frac{\omega^2}{k^2} + c_{44}\right) - (c_{13}+c_{44})^2,$$

$$\underline{B} = 4c_{11}c_{44}\left(\rho\frac{\omega^2}{k^2} + c_{33}\right)\left(\rho\frac{\omega^2}{k^2} + c_{44}\right)$$

Because of the generality of (4:41), the general solution of $\varphi(r)$ for the whole range of k is given, simply, as

$$\varphi(r) = k\left(A_1Z_n(pr) + B_1W_n(pr)\right)/(c_{13}+c_{44}) + A_2Z_n(qr) + B_2W_n(q_kr), \qquad (4:51)$$

where $p = p(R,\theta)$ and $q = (R,\theta)$ as given by (4:49).

CHAPTER V

1. Preliminaries of the Problem and Nondimensionalization of Equations

Consider a right circular cylindrical shell of infinite axial length (see Figure 5). Let the shell consist of three layers, each transversely isotropic and perfectly bonded together at their mutual interfaces. Let the inner material, bounded by the radii $r = a$, $r = b$, be referred to as material 1, and all equations, elastic constants, scalar potentials, etc., when written for material 1 be superscripted as "(1)". Similarly, for the layers bounded by $r = b$, $r = c$, and $r = c$, $r = d$, let the superscripts "(2)" and "(3)", respectively, be employed.

The axial wave number k and frequency ω of harmonic plane waves traveling in the axial direction along the cylinder will be the same for all layers since the layers are perfectly bonded. The solution of the anisotropic equations of motion, presented in Chapter II, is valid for each layer of the cylinder.

Before proceeding with the development of the frequency equation, all equations to be used will first be nondimensionalized. Executing the nondimensionalization at this stage will eliminate the necessity of developing a subsequent notation for the numerical analysis.

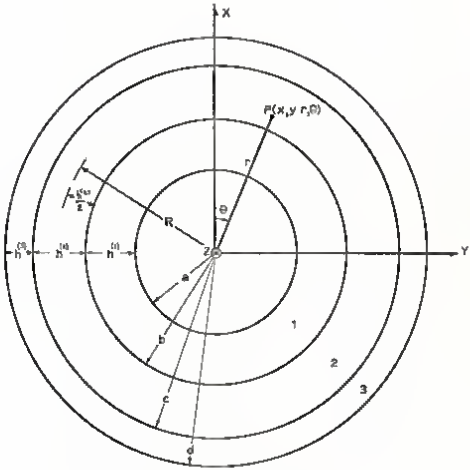

FIGURE 5. Cross-Section of Cylinder

For an infinite flat plate of thickness h, the shear velocity of the first antisymmetric thickness shear mode is $\sqrt{\mu/\rho}$, the frequency is $\sqrt{\mu}/h\sqrt{\rho}$ [72]. By analogy the frequency $\omega_s^{(2)}$, given by

$$\omega_s^{(2)} = \frac{\pi}{h^{(2)}}\sqrt{\frac{c_{44}^{(2)}}{\rho^{(2)}}}, \tag{5:1}$$

will be used to nondimensionalize ω. The term $\sqrt{c_{44}^{(2)}/\rho^{(2)}}$ is the velocity of equivoluminal waves traveling in the direction of the elastic symmetry axis of a transversely isotropic, infinite body [73] with the mechanical properties of layer 2 of the cylinder. Hence, (5:1) is the first mode thickness shear frequency for a plate of material 2 and thickness of the middle layer. In the limit, furthermore, as the constants pass to the isotropic values, (5:1) reduces exactly to $\sqrt{\mu/\rho}/h$. Using (5:1), the nondimensional frequency of the traveling waves is defined as

$$\Omega = \frac{\omega}{\omega_s^{(2)}} \tag{5:2}$$

The nondimensional wave number is defined as

$$\xi = \frac{k\,b^{(2)}}{\pi} = \frac{b^{(2)}}{L}, \tag{5:3}$$

where L is the half wavelength. The nondimensional variables Ω and ξ will be retained throughout the analysis as the dependent and independent variables, respectively.

Denote by $v_a^{(i)}$ the quantity

$$v_s^{(i)} = \sqrt{\frac{c_{44}^{(i)}}{\rho^{(i)}}}, \quad (i = 1,2,3), \tag{5:4}$$

and by $\delta^{(i)}$ the ratio

$$\delta^{(i)} = \frac{v_s^{(i)}}{v_s^{(2)}}, \quad (i = 1,2,3) \tag{5:5}$$

Denote by $v_d^{(i)}$ the quantity

$$V_d^{(i)} = \sqrt{\frac{c_{33}^{(i)}}{\rho^{(i)}}} \quad , \quad (i = 1,2,3), \tag{5:6}$$

and by $\eta^{(i)}$ the ratio

$$\eta^{(i)} = \frac{v_d^{(i)}}{v_s^{(i)}} \quad , \quad (i = 1,2,3) \tag{5:7}$$

The "velocity" v_d has no specific counterpart in the wave velocities in infinite bodies, but is named here only for convenience. In the limit, as the constants pass to the isotropic values, v_d approaches the velocity of dilatational waves in an infinite isotropic body.

For brevity denote the ratio Ω/ξ by

$$Z = \Omega/\xi \tag{5:8}$$

From (5:2) and (5:3) one obtains

$$Z = \frac{\omega}{\omega_s^{(i)}} \frac{\eta}{k\eta^{(i)}} = \frac{v}{v_s^{(i)}} . \tag{5:9}$$

Hence, Z is seen to be the nondimensional phase velocity of the traveling waves.

Finally, let

$$\begin{aligned} d_1^{(i)} &= \frac{c_{11}^{(i)}}{c_{44}^{(i)}} \quad , \quad d_2^{(i)} = \frac{c_{12}^{(i)}}{c_{44}^{(i)}} \quad , \quad d_3^{(i)} = \frac{c_{13}^{(i)}}{c_{44}^{(i)}} \quad , \\ d_4^{(i)} &= \frac{c_{44}^{(i)}}{c_{44}^{(i)}} \quad , \quad d_5^{(i)} = \frac{c_{55}^{(i)}}{c_{44}^{(i)}} \quad , \quad d_6^{(i)} = \frac{1}{2}(d_1^{(i)} - d_2^{(i)}), \end{aligned} \qquad (i = 1,2,3). \tag{5:10}$$

The radial wave numbers p^2 and q^2, on the basis of the above definitions, may be written as (see equation (3:31))

$$\begin{Bmatrix} p^{(i)2} \\ q^{(i)2} \end{Bmatrix} = \frac{k^2}{2 d_4^{(i)} d_5^{(i)}} \left\{ d_1^{(i)} d_5^{(i)} \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) + d_4^{(i)2} \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) + \left(d_5^{(i)} + d_4^{(i)} \right)^2 \pm \right.$$

$$\pm \left\{ \left[d_1^{(i)} d_5^{(i)} \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) + d_4^{(i)2} \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) + \left(d_5^{(i)} + d_4^{(i)} \right)^2 \right]^2 - \right.$$

$$\left. \left. - 4 d_1^{(i)} d_5^{(i)} d_4^{(i)2} \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) \left(\frac{Z^2}{\beta^{(i)2}} - 1 \right) \right\}^{1/2} \right\} .$$

Multiplying both sides of the above by $(h^{(0)}/\pi)^2$, using (5:3), and defining

$$P^{(0)2} = \left(\frac{h^{(0)}}{\pi}p\right)^2 \quad , \quad Q^{(0)2} = \left(\frac{h^{(0)}}{\pi}q\right)^2, \tag{5:11}$$

the radial wave numbers become, with the aid of (4:37), (4:38), and (4:39)

$$P^{(i)} = \sqrt{\epsilon_1^{(i)}}\,\gamma_P^{(i)}, \tag{a}$$
$$Q^{(i)} = \sqrt{\epsilon_2^{(i)}}\,\gamma_Q^{(i)}. \tag{b} \qquad (5:12)$$

The range of the sign factors $\epsilon_1^{(i)}$ and $\epsilon_2^{(i)}$, defined in (4:35) and (4:36), are given now as

$$\epsilon_1^{(i)} = \begin{cases} 1 & \text{if} \quad Z^2/b^{(i)2} > 1, \\ -1 & \text{if} \quad 1 > Z^2/b^{(i)2} \geq Z_{L1}^{(i)2}/b^{(i)2}, \\ 1 & \text{if} \quad Z_{L1}^{(i)2}/b^{(i)2} > Z^2/b^{(i)2} > Z_{L3}^{(i)2}/b^{(i)2}, \\ -1 & \text{if} \quad Z_{L3}^{(i)2}/b^{(i)2} \geq Z^2/b^{(i)2} \geq 0, \end{cases} \tag{5:13}$$

or

$$\epsilon_1^{(i)} = \begin{cases} 1 & \text{if} \quad Z^2/b^{(i)2} > Z_{L3}^{(i)2}/b^{(i)2}, \\ -1 & \text{if} \quad Z_{L3}^{(i)2}/b^{(i)2} \geq Z^2/b^{(i)2} \geq 0, \end{cases} \quad \text{cI.II (b)}$$

and

$$\epsilon_2^{(i)} = \begin{cases} 1 & \text{if} \quad Z^2/b^{(i)2} > d_3^{(i)}/d_4^{(i)}, \\ -1 & \text{if} \quad d_3^{(i)}/d_4^{(i)} > Z^2/b^{(i)2} \geq Z_{L1}^{(i)2}/b^{(i)2}, \\ 1 & \text{if} \quad Z_{L1}^{(i)2}/b^{(i)2} > Z^2/b^{(i)2} > Z_{L3}^{(i)2}/b^{(i)2}, \\ -1 & \text{if} \quad Z_{L3}^{(i)2}/b^{(i)2} \geq Z^2/b^{(i)2} \geq 0, \end{cases} \quad \text{cI.I (a)}$$

or

$$\epsilon_2^{(i)} = \begin{cases} 1 & \text{if} \quad Z^2/b^{(i)2} > d_3^{(i)}/d_4^{(i)}, \\ -1 & \text{if} \quad d_3^{(i)}/d_4^{(i)} > Z^2/b^{(i)2} > 1, \\ 1 & \text{if} \quad 1 > Z^2/b^{(i)2} > Z_{L3}^{(i)2}/b^{(i)2}, \\ -1 & \text{if} \quad Z_{L3}^{(i)2}/b^{(i)2} \geq Z^2/b^{(i)2} \geq 0 \end{cases} \quad \text{cI.II (b)} \qquad (5:14)$$

In the range equations (5:13) and (5:14), the numbers

$Z_{L1}^{(0)2}/b^{(0)2}$ and $Z_{L3}^{(0)2}/b^{(0)2}$ are found from (4:14) to be

$$\frac{Z_{L1}^{(0)2}}{b^{(0)2}}\ ,\ \frac{Z_{L3}^{(0)2}}{b^{(0)2}} = \frac{1}{d_4^{(0)}}\left\{\frac{(d_3^{(0)}d_3^{(0)}-d_4^{(0)2})(d_1^{(0)}-d_4^{(0)})-(d_1^{(0)}d_4^{(0)}\mp d_3^{(0)}\sqrt{d_4^{(0)2}+d_3^{(0)2}})}{(d_1^{(0)}-d_4^{(0)})^2}\pm\right.$$

$$\left.\pm\frac{2(d_3^{(0)}+d_4^{(0)})}{(d_1^{(0)}-d_4^{(0)})^2}\sqrt{d_1^{(0)}d_4^{(0)}\left[(d_3^{(0)}+d_4^{(0)})^2-(d_1^{(0)}-d_4^{(0)})(d_3^{(0)}-d_4^{(0)})\right]}\right\}. \qquad (5:15)$$

Now, based on the rangs equations (5:13) and (5:14), $\underline{\beta}^{(0)}$ and $\underline{\alpha}^{(0)}$ are given as

(a) (5:16)

$$\left.\begin{matrix}\underline{\beta}^{(0)}\\ \underline{\alpha}^{(0)}\end{matrix}\right\} = \sqrt{\frac{1}{2}}\left\{\frac{d_3^{(0)}}{d_4^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_4^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_1^{(0)}}{d_4^{(0)}}\left(\frac{d_3^{(0)}}{d_4^{(0)}}+1\right)^2\pm\right.$$

$$\left.\pm\sqrt{\left[\frac{d_3^{(0)}}{d_4^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_4^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_1^{(0)}}{d_4^{(0)}}\left(\frac{d_3^{(0)}}{d_4^{(0)}}+1\right)^2\right]^2-4\,\frac{d_3^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)\left(\frac{Z^2}{b^{(0)2}}-1\right)}\right\}^{1/2}$$

if $Z^2/b^{(0)2}$ is contained in $[0,Z_{L3}^{(0)2}/b^{(0)2}]$ or $[Z_{L1}^{(0)2}/b^{(0)2},\infty)$; or

(b) (5:16)

$$\left.\begin{matrix}\underline{\beta}^{(0)}\\ \underline{\alpha}^{(0)}\end{matrix}\right\} = \underline{\beta}^{(0)1/4}\left(\pm\cos\underline{\theta}^{(0)}+\imath\sin\underline{\theta}^{(0)}\right),$$

where

$$\underline{\beta}^{(0)1/4} = \left|\frac{d_3^{(0)}}{d_4^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)\left(\frac{Z^2}{b^{(0)2}}-1\right)\right|^{1/4}, \qquad (5:17)$$

$$\underline{\theta}^{(0)} = \frac{1}{2}\tan^{-1}\left\{\frac{\left|4\,\underline{\beta}^{(0)}-\left[\frac{d_3^{(0)}}{d_4^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_4^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_1^{(0)}}{d_4^{(0)}}\left(\frac{d_3^{(0)}}{d_4^{(0)}}+1\right)^2\right]^2\right|^{1/2}}{\frac{d_3^{(0)}}{d_4^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_4^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right)+\frac{d_1^{(0)}}{d_4^{(0)}}\left(\frac{d_3^{(0)}}{d_4^{(0)}}+1\right)^2}\right\}$$

if $Z^2/b^{(0)2}$ is contained in $(Z_{L3}^{(0)2}/b^{(0)2},Z_{L1}^{(0)2}/b^{(0)2})$.

Turning now to the radial wave number s, the nondimensionalization is performed as for p and q. Hence

$$\underline{s}^{(0)2} = \left(\frac{h^{(0)}s^{(0)}}{\pi}\right)^2 = \zeta\,\frac{d_4^{(0)}}{d_3^{(0)}}\left(\frac{Z^2}{b^{(0)2}}-1\right). \qquad (5:18)$$

Using equations (4:2) and (4:3) alters (5:18) to the form

$$\underline{s}^{(0)2} = \sqrt{\varepsilon_3^{(0)}}\,\zeta\,\underline{s}^{(0)}, \qquad (5:19)$$

where

$$\epsilon_3^{(i)} = \begin{cases} 1 & \text{if } \bar{z}^2/b^{(i)2} > 1, \\ -1 & \text{if } 1 > \bar{z}^2/b^{(i)2} \geq 0, \end{cases} \qquad \text{(a)}$$

and (5:20)

$$\xi^{(i)} = \sqrt{\frac{d_4^{(i)}}{d_5^{(i)}}} \left| \frac{\bar{z}^2}{b^{(i)2}} - 1 \right|^{1/2}. \qquad \text{(b)}$$

Writing equations (3:37) as

$$\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{n^2}{\bar{r}^2} + \beta^{(i)2}\right)\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{n^2}{\bar{r}^2} + \varrho^{(i)2}\right)\varphi^{(i)}(\bar{r}) = 0, \qquad \text{(a)}$$

and (5:21)

$$\left(\frac{d^2}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d}{d\bar{r}} - \frac{n^2}{\bar{r}^2} + \varrho^{(i)2}\right)f^{(i)}(\bar{r}) = 0, \qquad \text{(b)}$$

with

$$\bar{r} = \frac{r\,\eta}{h^{(i)}}, \qquad (5:22)$$

the displacement potentials assume the form

$$\varphi^{(i)}(\bar{r}) = \frac{\xi}{(\varrho_s^{(i)2} - \beta^{(i)2})} \frac{\eta}{h^{(i)}} \frac{1}{c_{44}^{(i)}} \left(\bar{A}_1^{(i)} Z_n(\gamma_s \varrho^{(i)}\bar{r}) + \bar{B}_1^{(i)} W_n(\gamma_s \varrho^{(i)}\bar{r})\right) + \bar{A}_2^{(i)} Z_n(\gamma_s \beta^{(i)}\bar{r}) + \bar{B}_2^{(i)} W_n(\gamma_s \beta^{(i)}\bar{r}), \qquad (5:23)$$

$$f^{(i)}(\bar{r}) = \bar{A}_3^{(i)} Z_n(\gamma_s \varrho^{(i)}\bar{r}) + \bar{B}_3^{(i)} W_n(\gamma_s \varrho^{(i)}\bar{r}).$$

Using the above nondimensionalized equations, the displacement field (3:56) can be written as

$$u_r^{(i)} = \frac{\eta}{h^{(i)}}\left[\frac{\eta}{h^{(i)}}\gamma_s \frac{d\varphi^{(i)}}{d\bar{r}} + c_{44}^{(i)}\left(d_6^{(i)} d_4^{(i)} \frac{d\varphi_2^{(i)}}{d\bar{r}} + \frac{n}{\bar{r}} f^{(i)}\right)\right] \cos n\theta \cos(\gamma_s \bar{z} - \Omega\bar{t}),$$

$$u_\theta^{(i)} = -\frac{n}{h^{(i)}}\left[\frac{\eta}{h^{(i)}}\gamma_s \frac{n}{\bar{r}} \varphi_1^{(i)} + c_{44}^{(i)}\left(d_5^{(i)} d_4^{(i)}\right)\frac{\eta}{\bar{r}} \varphi_2^{(i)} + \frac{1}{d}\frac{df^{(i)}}{d\bar{r}}\right] \sin n\theta \cos(\gamma_s \bar{z} - \Omega\bar{t}), \qquad (5:24)$$

$$u_z^{(i)} = \frac{\eta}{h^{(i)}}\left[\frac{\eta}{h^{(i)}}\gamma_s^2\left(d_1^{(i)} \varepsilon_3^{(i)} \xi^{(i)2} - d_4^{(i)} \frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\right)\frac{\varphi_1^{(i)}}{d_5^{(i)} + d_4^{(i)}}\omega + \right.$$

$$\left. + c_{44}^{(i)}\gamma_s\left(d_1^{(i)} \varepsilon_3^{(i)} \xi^{(i)2} \frac{\bar{z}^2}{b^{(i)2}} - d_4^{(i)} \frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\right) \varphi_2^{(i)}\right] \cos n\theta \sin(\gamma_s \bar{z} - \Omega\bar{t}),$$

where

$$\bar{z} = \frac{z\,\eta}{h^{(i)}}, \quad \bar{t} = \omega_s^{(i)} t. \qquad (5:25)$$

is (5:24) the formulation $\varphi_1^{(i)}(\bar{r}) = k\varphi_1^{(i)}/(c_{44}^{(i)} + c_{13}^{(i)}) + \varphi_2^{(i)}$

was used along with

$$\nabla_1^2\varphi_1^{(i)}(\bar{r}) = -\frac{\rho\omega^2(1-\nu^{(i)2})}{c_{44}^{(i)}(c_{11}^{(i)}+c_{44}^{(i)})}\frac{R}{h^{(i)}}\varphi_1^{(i)}(\bar{r}) - \xi^2\varepsilon_1^{(i)}\beta^{(i)2}\varphi_2^{(i)}(\bar{r}). \qquad (5:26)$$

2. Derivation of the Frequency Equation

In order to complete the solution of the displacement field it is necessary to evaluate the integration constants $A_j^{(i)}$, $B_j^{(i)}$ $(i,j = 1,2,3)$ contained in the displacement potentials $\varphi(\bar{r})$ and $f(\bar{r})$. The boundary and interface conditions of the cylinder are used to establish relationships between the integration constants.

For a traction free cylinder the surface stresses must vanish along the inner and outer surfaces, that is

$$\tau_{r_j}^{(i)} = 0 \quad \text{at } \bar{r} = \bar{a}, \quad (j = r,\theta,z),$$

and (5:27)

$$\tau_{r_j}^{(3)} = 0 \quad \text{at } \bar{r} = \bar{c}, \quad (j = r,\theta,z)$$

The interface, or continuity, conditions for a cylinder with perfectly bonded layers are

$$u_j^{(i)} = u_j^{(i)} \text{ and } \tau_{r_j}^{(i)} = \tau_{r_j}^{(i)} \quad \text{at } \bar{r} = \bar{b}, (j = r,\theta,z),$$
$$u_j^{(2)} = u_j^{(3)} \text{ and } \tau_{r_j}^{(2)} = \tau_{r_j}^{(3)} \quad \text{at } \bar{r} = \bar{c}, (j = r,\theta,z). \qquad (5:28)$$

in these equations the nondimensional radii of the cylinder layers have been indicated by a bar, and are defined in (5:22) with r replaced by a, b, c, or d.

In order to employ these boundary and interface conditions the stress components must be expressed in terms of φ

and f. Using equations (2:14) for $\tau_{rj}^{(i)}$ (j = r,θ,z), the atrasses take the form

$$\tau_{rr}^{(i)} = c_{44}^{(i)} \frac{\eta}{k^{(i)}} \left[d_1^{(i)} \frac{\partial \varphi_r^{(i)}}{\partial \bar{r}} + d_2^{(i)} \left(\frac{1}{\bar{r}} \frac{\partial u_\theta^{(i)}}{\partial \theta} + \frac{u_r^{(i)}}{\bar{r}} \right) + d_5^{(i)} \frac{\partial u_z^{(i)}}{\partial \bar{z}} \right],$$

$$\tau_{r\theta}^{(i)} = c_{44}^{(i)} \frac{\eta}{k^{(i)}} d_6^{(i)} \left[\frac{\partial u_\theta^{(i)}}{\partial \bar{r}} - \frac{u_\theta^{(i)}}{\bar{r}} + \frac{1}{\bar{r}} \frac{\partial u_r^{(i)}}{\partial \theta} \right],$$

$$\tau_{rz}^{(i)} = c_{44}^{(i)} \frac{\eta}{k^{(i)}} d_4^{(i)} \left[\frac{\partial u_z^{(i)}}{\partial \bar{r}} + \frac{\partial u_r^{(i)}}{\partial \bar{z}} \right].$$

Substituting for the $u_j^{(i)}$ (j = r,θ,z) from equations (5:24), the atrasses become

$$\tau_{rr}^{(i)} = c_{44}^{(i)} \left(\frac{\eta}{k^{(i)}} \right)^2 \left\{ \frac{\eta}{k^{(i)}} \bar{r} \left[-2 d_6^{(i)} \frac{1}{\bar{r}} \frac{d}{d\bar{r}} + 2 d_6^{(i)} \frac{n^2}{\bar{r}^2} - \frac{d_4^{(i)}}{d_5^{(i)} + d_6^{(i)}} \left(d_1^{(i)} \frac{n^2}{\bar{r}^2} + d_5^{(i)} \bar{r}^2 \frac{\bar{z}^2}{k^{(i)2}} - d_5^{(i)} \bar{r}^2 \right) \right] \varphi_1^{(i)} + \right.$$

$$+ c_{44}^{(i)} \left(d_5^{(i)} + d_6^{(i)} \right) \left[-2 d_6^{(i)} \frac{1}{\bar{r}} \frac{d}{d\bar{r}} + 2 d_6^{(i)} \frac{n^2}{\bar{r}^2} - \frac{d_4^{(i)}}{d_5^{(i)} + d_6^{(i)}} \left(d_1^{(i)} Q^{(i)2} + d_5^{(i)} \bar{r}^2 \frac{\bar{z}^2}{k^{(i)2}} - d_5^{(i)} \bar{r}^2 \right) \right] \varphi_2^{(i)} +$$

$$\left. + 2 n d_6^{(i)} \left[\frac{1}{\bar{r}} \frac{d}{d\bar{r}} - \frac{1}{\bar{r}^2} \right] f^{(i)} \right\} \cos n\theta \cos(\bar{r}\bar{z} - \Omega\bar{t}), \qquad \text{(a)}$$

$$\tau_{r\theta}^{(i)} = c_{44}^{(i)} \left(\frac{\eta}{k^{(i)}} \right)^2 \left[2 \frac{\eta}{k^{(i)}} \bar{r} d_6^{(i)} \left[-\frac{n}{\bar{r}} \frac{d}{d\bar{r}} + \frac{n}{\bar{r}^2} \right] \varphi_1^{(i)} + \right.$$

$$+ 2 c_{44}^{(i)} \left(d_5^{(i)} + d_4^{(i)} \right) d_6^{(i)} \left[-\frac{n}{\bar{r}} \frac{d}{d\bar{r}} + \frac{n}{\bar{r}^2} \right] \varphi_2^{(i)} + \qquad (5:29)$$

$$\left. + d_6^{(i)} \left[\frac{2}{\bar{r}} \frac{d}{d\bar{r}} - 2 \frac{n^2}{\bar{r}^2} + \bar{\zeta}^{(i)2} \right] f^{(i)} \right\} \sin n\theta \cos(\bar{r}\bar{z} - \Omega\bar{t}), \qquad \text{(b)}$$

$$\tau_{rz}^{(i)} = c_{44}^{(i)} \left(\frac{\eta}{k^{(i)}} \right)^2 \left\{ \frac{\eta}{k^{(i)}} \bar{r} \left[\frac{d_4^{(i)}}{d_5^{(i)} + d_6^{(i)}} \left(d_1^{(i)} Q^{(i)2} - d_5^{(i)} \bar{r}^2 - d_4^{(i)} \bar{r}^2 \frac{\bar{z}^2}{k^{(i)2}} \right) \frac{d\varphi_1^{(i)}}{d\bar{r}} + \right.\right.$$

$$\left.\left. + c_{44}^{(i)} \frac{d_4^{(i)}}{\bar{r}} \left(d_1^{(i)} Q^{(i)2} - d_5^{(i)} \bar{r}^2 - d_4^{(i)} \bar{r}^2 \frac{\bar{z}^2}{k^{(i)2}} \right) \frac{d\varphi_2^{(i)}}{d\bar{r}} - \frac{n}{\bar{r}} \bar{r} f^{(i)} \right] \cos n\theta \sin(\bar{r}\bar{z} - \Omega\bar{t}) \right\}. \qquad \text{(c)}$$

Consideration of equations (5:24) indicate that the integration constants of the three potential functions must have the following dimensions:

$$\left.\begin{matrix} \bar{A}_1^{(i)} \\ \bar{B}_1^{(i)} \end{matrix}\right\} = [L^3], \qquad \left.\begin{matrix} \bar{A}_2^{(i)} \\ \bar{B}_2^{(i)} \end{matrix}\right\} = \left[\frac{L^4}{T}\right], \qquad \left.\begin{matrix} \bar{A}_3^{(i)} \\ \bar{B}_3^{(i)} \end{matrix}\right\} = [L^2], \qquad (5:30)$$

where L = length and F = force. Nondimensionalizing the integration constants so that the displacement field retains the dimension of length yields

$$\begin{Bmatrix} A_1^{(i)} \\ B_1^{(i)} \end{Bmatrix} = \left(\frac{\pi}{\bar{h}^{(i)}}\right)^3 \begin{Bmatrix} \bar{A}_1^{(i)} \\ \bar{B}_1^{(i)} \end{Bmatrix}, \quad \begin{Bmatrix} A_2^{(i)} \\ B_2^{(i)} \end{Bmatrix} = C_{44}^{(i)} \left(\frac{\pi}{\bar{h}^{(i)}}\right)^2 \begin{Bmatrix} \bar{A}_2^{(i)} \\ \bar{B}_2^{(i)} \end{Bmatrix}, \quad \begin{Bmatrix} A_3^{(i)} \\ B_3^{(i)} \end{Bmatrix} = \left(\frac{\pi}{\bar{h}^{(i)}}\right)^2 \begin{Bmatrix} \bar{A}_3^{(i)} \\ \bar{B}_3^{(i)} \end{Bmatrix} \quad (5:31)$$

In order to employ equations (5:23) for $\varphi^{(i)}$ and $f^{(i)}$ in equations (5:24) and (5:29), differentiation formulas for Z_n and W_n must be developed. Consider Z_n initially, and suppose the argument of this function is $\gamma\bar{r}$, where γ may be real, imaginary, or complex. Z_n may represent either a regular of modified Bessel function of the first kind, hence

$$\frac{d}{d\bar{r}}\left[J_n(\gamma\bar{r})\right] = -\gamma J_{n+1}(\gamma\bar{r}) + \frac{n}{\bar{r}} J_n(\gamma\bar{r}),$$

and

$$\frac{d}{d\bar{r}}\left[I_n(|\gamma|\bar{r})\right] = |\gamma| I_{n+1}(|\gamma|\bar{r}) + \frac{n}{\bar{r}} I_n(|\gamma|\bar{r})$$

Since the modified Bessel function derives from the regular function when the argument is imaginary, it follows that the following differentiation formula

$$\frac{d}{d\bar{r}}\left[Z_n(\gamma\bar{r})\right] = -\epsilon\gamma Z_{n+1}(\gamma\bar{r}) + \frac{n}{\bar{r}} Z_n(\gamma\bar{r}) \qquad \text{(a) (5:32)}$$

holds for either $J_n(\gamma\bar{r})$ or $I_n(\gamma\bar{r})$ if γ is understood to be the modulus if the argument is pure imaginary, and if, further, ϵ is a sign factor of the type previously employed for $\underline{p}^{(i)}$, $\underline{q}^{(i)}$, and $\underline{s}^{(i)}$; which is -1 only when the argument is imaginary. In a similar fashion it follows that

$$\frac{d}{d\bar{r}}\left[W_n(\gamma\bar{r})\right] = -\gamma W_{n+1}(\gamma\bar{r}) + \frac{n}{\bar{r}} W_n(\gamma\bar{r}), \qquad \text{(b) (5:32)}$$

since the differentiation formulas for $Y_n(\gamma\bar{r})$ and $K_n(\gamma\bar{r})$ are

identical, and γ is understood to be the modulus when the argument is imaginary.

Substituting equations (5:23) into (5:24) and (5:29), making use of (5:31), and using (5:32) where necessary, the following expressions for the displacement and stress components are obtained:

$$u_r^{(i)} = \frac{k^{(i)}}{\pi\bar{r}}\Big\{A_1^{(i)}\gamma\Big[-\varepsilon_1^{(i)}\gamma\alpha^{(i)}\bar{r}\,Z_{n+1}(\gamma\alpha^{(i)}\bar{r}) + nZ_n(\gamma\alpha^{(i)}\bar{r})\Big] +$$
$$+ B_1^{(i)}\gamma\Big[-\gamma\beta^{(i)}\bar{r}\,W_{n+1}(\gamma\beta^{(i)}\bar{r}) + nW_n(\gamma\beta^{(i)}\bar{r})\Big] +$$
$$+ A_2^{(i)}(d_3^{(i)}+d_4^{(i)})\Big[-\varepsilon_2^{(i)}\gamma\beta^{(i)}\bar{r}\,Z_{n+1}(\gamma\beta^{(i)}\bar{r}) + nZ_n(\gamma\beta^{(i)}\bar{r})\Big] +$$
$$+ B_2^{(i)}(d_3^{(i)}+d_4^{(i)})\Big[-\gamma\beta^{(i)}\bar{r}\,W_{n+1}(\gamma\beta^{(i)}\bar{r}) + nW_n(\gamma\beta^{(i)}\bar{r})\Big] +$$
$$+ A_3^{(i)}nZ_n(\gamma\xi^{(i)}\bar{r}) + B_3^{(i)}nW_n(\gamma\xi^{(i)}\bar{r})\Big\}\cos n\theta\,\cos(\gamma\bar{z}-\Omega\bar{t}), \quad (a)$$

$$u_\theta^{(i)} = \frac{k^{(i)}}{\pi\bar{r}}\Big\{-A_1^{(i)}n\gamma Z_n(\gamma\alpha^{(i)}\bar{r}) - B_1^{(i)}n\gamma W_n(\gamma\alpha^{(i)}\bar{r}) -$$
$$- A_2^{(i)}n(d_3^{(i)}+d_4^{(i)})Z_n(\gamma\beta^{(i)}\bar{r}) - B_2^{(i)}n(d_3^{(i)}+d_4^{(i)})W_n(\gamma\beta^{(i)}\bar{r}) -$$
$$- A_3^{(i)}\Big[-\varepsilon_3^{(i)}\gamma\xi^{(i)}\bar{r}\,Z_{n+1}(\gamma\xi^{(i)}\bar{r}) + nZ_n(\gamma\xi^{(i)}\bar{r})\Big] - \quad (5:33)$$
$$- B_3^{(i)}\Big[-\gamma\xi^{(i)}\bar{r}\,W_{n+1}(\gamma\xi^{(i)}\bar{r}) + nW_n(\gamma\xi^{(i)}\bar{r})\Big]\Big\}\sin n\theta\,\cos(\gamma\bar{z}-\Omega\bar{t}), \quad (b)$$

$$u_z^{(i)} = \frac{k^{(i)}}{\pi\bar{r}}\Big\{A_1^{(i)}\frac{\gamma^2\bar{r}}{d_3^{(i)}+d_4^{(i)}}\Big(d_1^{(i)}\varepsilon_1^{(i)}\alpha^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\Big)Z_n(\gamma\alpha^{(i)}\bar{r}) +$$
$$+ B_1^{(i)}\frac{\gamma^2\bar{r}}{d_3^{(i)}+d_4^{(i)}}\Big(d_1^{(i)}\varepsilon_1^{(i)}\alpha^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\Big)W_n(\gamma\alpha^{(i)}\bar{r}) +$$
$$+ A_2^{(i)}\gamma\bar{r}\Big(d_1^{(i)}\varepsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\Big)Z_n(\gamma\beta^{(i)}\bar{r}) + \quad (c)$$
$$+ B_2^{(i)}\gamma\bar{r}\Big(d_1^{(i)}\varepsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{b^{(i)2}} + d_4^{(i)}\Big)W_n(\gamma\beta^{(i)}\bar{r})\Big\}\cos n\theta\,\sin(\gamma\bar{z}-\Omega\bar{t}),$$

$$\tau_{rr}^{(i)} = \frac{c_{44}^{(i)}}{\bar{r}^2}\Big\{A_1^{(i)}\zeta\Big[2d_6^{(i)}\epsilon_1^{(i)}\zeta\,\omega^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\omega^{(i)}\bar{r}) + \Big\{2d_6^{(i)}n(n-1) - \frac{d_5^{(i)}\zeta^2\bar{r}^2}{d_5^{(i)}+d_6^{(i)}}\Big[d_1^{(i)}\epsilon_1^{(i)}\omega^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\bar{r}^2}\omega^{(i)2} - d_5^{(i)}\Big]\Big\}Z_n(\zeta\,\omega^{(i)}\bar{r})\Big] + B_1^{(i)}\zeta\Big[2d_6^{(i)}\zeta\,\omega^{(i)}\bar{r}\,W_{n+1}(\zeta\,\omega^{(i)}\bar{r}) + \Big\{2d_6^{(i)}n(n-1) - \frac{d_5^{(i)}\zeta^2\bar{r}^2}{d_5^{(i)}+d_6^{(i)}}\Big[d_1^{(i)}\epsilon_1^{(i)}\omega^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\bar{r}^2}\omega^{(i)2} - d_5^{(i)}\Big]\Big\}W_n(\zeta\,\omega^{(i)}\bar{r})\Big] + A_2^{(i)}(d_5^{(i)}+d_6^{(i)})\Big[2d_6^{(i)}\epsilon_2^{(i)}\zeta\,\beta^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\beta^{(i)}\bar{r}) + \Big\{2d_6^{(i)}n(n-1) - \frac{d_5^{(i)}\zeta^2\bar{r}^2}{d_5^{(i)}+d_6^{(i)}}\Big[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\bar{r}^2}\beta^{(i)2} - d_5^{(i)}\Big]\Big\}Z_n(\zeta\,\beta^{(i)}\bar{r})\Big] + B_2^{(i)}(d_5^{(i)}+d_6^{(i)})\Big[2d_6^{(i)}\zeta\,\beta^{(i)}\bar{r}\,W_{n+1}(\zeta\,\beta^{(i)}\bar{r}) + \Big\{2d_6^{(i)}n(n-1) - \frac{d_5^{(i)}\zeta^2\bar{r}^2}{d_5^{(i)}+d_6^{(i)}}\Big[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\bar{r}^2}\beta^{(i)2} - d_5^{(i)}\Big]\Big\}W_n(\zeta\,\beta^{(i)}\bar{r})\Big] + A_3^{(i)}2n\,d_6^{(i)}\Big[-\epsilon_3^{(i)}\zeta\,\xi^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\xi^{(i)}\bar{r}) + (n-1)Z_n(\zeta\,\xi^{(i)}\bar{r})\Big] + \qquad (d)$$

$$+ B_3^{(i)}2n\,d_6^{(i)}\Big[-\zeta\,\xi^{(i)}\bar{r}\,W_{n+1}(\zeta\,\xi^{(i)}\bar{r}) + (n-1)W_n(\zeta\,\xi^{(i)}\bar{r})\Big]\Big\}\cos n\theta\cos(\zeta\bar{z}-\Omega\bar{t}),$$

(5:33) (cont.)

$$\tau_{r\theta}^{(i)} = \frac{c_{44}^{(i)}}{\bar{r}^2}\Big\{A_1^{(i)}2n\,d_6^{(i)}\zeta\Big[\epsilon_1^{(i)}\zeta\,\omega^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\omega^{(i)}\bar{r}) - (n-1)Z_n(\zeta\,\omega^{(i)}\bar{r})\Big] + B_1^{(i)}2n\,d_6^{(i)}\zeta\Big[\zeta\,\omega^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\omega^{(i)}\bar{r}) - (n-1)W_n(\zeta\,\omega^{(i)}\bar{r})\Big] + A_2^{(i)}(d_5^{(i)}+d_6^{(i)})2n\,d_6^{(i)}\zeta\Big[\epsilon_2^{(i)}\zeta\,\beta^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\beta^{(i)}\bar{r}) - (n-1)Z_n(\zeta\,\beta^{(i)}\bar{r})\Big] + B_2^{(i)}(d_5^{(i)}+d_6^{(i)})2n\,d_6^{(i)}\zeta\Big[\zeta\,\beta^{(i)}\bar{r}\,W_{n+1}(\zeta\,\beta^{(i)}\bar{r}) - (n-1)W_n(\zeta\,\beta^{(i)}\bar{r})\Big] + \qquad (e)$$

$$+ A_3^{(i)}d_6^{(i)}\Big[-2\epsilon_3^{(i)}\zeta\,\xi^{(i)}\bar{r}\,Z_{n+1}(\zeta\,\xi^{(i)}\bar{r}) - \Big(2n(n-1) - \epsilon_3^{(i)}\zeta^2\,\xi^{(i)2}\bar{r}^2\Big)Z_n(\zeta\,\xi^{(i)}\bar{r})\Big]$$

$$+ B_3^{(i)}d_6^{(i)}\Big[-2\zeta\,\xi^{(i)}\bar{r}\,W_{n+1}(\zeta\,\xi^{(i)}\bar{r}) - \Big(2n(n-1) - \epsilon_3^{(i)}\zeta^2\,\xi^{(i)2}\bar{r}^2\Big)W_n(\zeta\,\xi^{(i)}\bar{r})\Big]\Big\}\times$$

$$\times\sin n\theta\cos(\zeta\bar{z}-\Omega\bar{t}),$$

$$\tau_{rz}^{(i)} = \frac{c_{44}^{(i)}}{h^2}\Big\{A_1^{(i)}\frac{d_3^{(i)}\xi_r^{(i)}}{d_3^{(i)}+d_4^{(i)}}\Big[\varepsilon_1^{(i)}\alpha^{(i)2} - d_5^{(i)}\frac{Z^2}{\beta_{(i)2}} - d_5^{(i)}\Big]\Big[-\xi_r^{(i)}\zeta_\alpha^{(i)}\mp Z_{n+1}(\zeta_\alpha^{(i)}\bar{r}) + n Z_n(\zeta_\alpha^{(i)}\bar{r})\Big] +$$

$$+ B_1^{(i)}\frac{d_3^{(i)}\xi_r^{(i)}}{d_3^{(i)}+d_4^{(i)}}\Big[d_1^{(i)}\varepsilon_1^{(i)}\alpha^{(i)2} - d_4^{(i)}\frac{Z^2}{\beta_{(i)2}} - d_5^{(i)}\Big]\Big[-\xi_r\alpha^{(i)}\mp W_{n+1}(\zeta_\alpha^{(i)}\bar{r}) + n W_n(\zeta_\alpha^{(i)}\bar{r})\Big] +$$

$$+ A_2^{(i)}d_4^{(i)}\zeta_r\mp\Big[d_1^{(i)}\varepsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\beta_{(i)2}} - d_5^{(i)}\Big]\Big[-\varepsilon_2^{(i)}\zeta_\beta^{(i)}\mp Z_{n+1}(\zeta_\beta^{(i)}\bar{r}) + n Z_n(\zeta_\beta^{(i)}\bar{r})\Big] +$$

$$+ B_2^{(i)}d_4^{(i)}\zeta_r\mp\Big[d_1^{(i)}\varepsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\beta_{(i)2}} - d_5^{(i)}\Big]\Big[\zeta_\beta^{(i)}\mp W_{n+1}(\zeta_\beta^{(i)}\bar{r}) + n W_n(\zeta_\beta^{(i)}\bar{r})\Big] -$$

$$- A_3^{(i)}n\zeta_r\mp Z_n(\zeta_\alpha^{(i)}\bar{r}) - B_3^{(i)}n\zeta_r\mp W_n(\zeta_\beta^{(i)}\bar{r})\Big\}\cos n\theta\sin(\zeta_z - \Omega t). \qquad \text{(f)} \quad \begin{matrix}(5:33)\\(\text{cont})\end{matrix}$$

Equations (5:33) are applied in the boundary and interface condition equations (5:27) and (5:28) to form a system of 18 homogeneous equations in the 18 integration constants $A_j^{(i)}$, $B_j^{(i)}$ $(i,j = 1,2,3)$. In order that nontrivial solutions for the integration constants be obtained, it is necessary (but not always sufficient) that the coefficient determinant of the integration constants vanish, resulting in the equation

$$\det|C_{ij}| = 0, \quad (i,j = 1,2, \ldots, 18) \qquad (5:34)$$

This equation is the so called frequency or dispersion equation.

As mentioned above, the vanishing of (5:34) is not always sufficient to assure nontrivial solutions of the integration constants. Equations (5:33) are not always the appropriate solutions for the stress and displacement equations as was revealed in Chapter IV. Thus, when $Z^2/b^{(i)2}$ attains the value of a turning point number for $\underline{p}^{(i)}$, $\underline{q}^{(i)}$, or $\underline{s}^{(i)}$, or when $Z^2/b^{(i)2}$ attains such a value that one of the three pairs of $\underline{p}^{(i)}$ and $\underline{q}^{(i)}$ are equal, equations (5:33) fail to represent the displacement and stress field components. To this extent, equation (5:34) is insufficient. When viewed in the broader light

of the general and particular solutions, (5:34) is the only condition imposable to obtain nontrivial solutions.

In ensuing discussions it will be necessary to understand the construction of the determinant (5:34) in some detail. The rows of (5:34) will be written from the boundary and interface conditions arranged in the following order:

row	condition	
1	$\tau_{rr}^{(1)} = 0$	$r = a$,
2	$\tau_{r\theta}^{(1)} = 0$	
3	$\tau_{rz}^{(1)} = 0$	
4	$\tau_{rz}^{(1)} - \tau_{rz}^{(2)} = 0$	$r = b$,
5	$\tau_{r\theta}^{(1)} - \tau_{r\theta}^{(2)} = 0$	
6	$\tau_{rr}^{(1)} - \tau_{rr}^{(2)} = 0$	
7	$u_r^{(1)} - u_r^{(2)} = 0$	
8	$u_\theta^{(1)} - u_\theta^{(2)} = 0$	
9	$u_z^{(1)} - u_z^{(2)} = 0$	
10	$-u_z^{(2)} + u_z^{(3)} = 0$	$r = c$,
11	$-u_\theta^{(2)} + u_\theta^{(3)} = 0$	
12	$-u_r^{(2)} + u_r^{(3)} = 0$	
13	$-\tau_{rr}^{(2)} + \tau_{rr}^{(3)} = 0$	
14	$-\tau_{r\theta}^{(2)} + \tau_{r\theta}^{(3)} = 0$	
15	$-\tau_{rz}^{(2)} + \tau_{rz}^{(3)} = 0$	
16	$\tau_{rz}^{(3)} = 0$	$r = d$
17	$\tau_{r\theta}^{(3)} = 0$	
18	$\tau_{rr}^{(3)} = 0$	

(5:35)

The columns of (5:34) will be written for the 18 integration constants arranged in the (logical) sequence

column: 1 2 3 4 5 6 7 8 9 10 11 12

constant: $A_1^{(1)}$ $B_1^{(1)}$ $A_2^{(1)}$ $B_2^{(1)}$ $A_3^{(1)}$ $B_3^{(1)}$ $A_1^{(2)}$ $B_1^{(2)}$ $A_2^{(2)}$ $B_2^{(2)}$ $A_3^{(2)}$ $B_3^{(2)}$

column(cont): 13 14 15 16 17 18 (5:36)

constant(cont): $A_1^{(3)}$ $B_1^{(3)}$ $A_2^{(3)}$ $B_2^{(3)}$ $A_3^{(3)}$ $B_3^{(3)}$

Superficial observations on the construction of (5:34) show that (a) a given column contains Bessel functions of only one kind, (b) only one radial wave number occurs in the argument of the Bessel functions of a given column, (c) 152 of the C_{ij} in (5:34) are zero and the remaining 172 are formed from equations (5:33) by adjustments on only the sign of the coefficient, the value of $\bar{r}$, and the superscripts.

The nonzero C_{ij} are listed below. The standard convention: i = row number and j = column number is employed, and these initial row and column numbers will be retained through the remainder of the work to preserve the identity of the elements.

$$C_{11} = 2d_6^{(1)}\epsilon_1^{(1)}\gamma^2\alpha^{(1)}\bar{\alpha}\,Z_{n+1}(\gamma\alpha^{(1)}\bar{\alpha}) + \gamma\left\{2d_6^{(1)}n(n-1) - \right.$$

$$\left. - \frac{d_4^{(1)}\gamma^2\bar{\alpha}^2}{d_5^{(1)}+d_4^{(1)}}\left[d_1^{(1)}\epsilon_1^{(1)}\alpha^{(1)2} + d_5^{(1)}\frac{\bar{r}^2}{\bar{\alpha}^{(1)2}} - d_5^{(1)}\right]\right\}Z_n(\gamma\alpha^{(1)}\bar{\alpha}),$$

$$C_{12} = 2d_6^{(1)}\gamma^2\alpha^{(1)}\bar{\alpha}\,W_{n+1}(\gamma\alpha^{(1)}\bar{\alpha}) + \gamma\left\{2d_6^{(1)}n(n-1) - \right.$$

$$\left. - \frac{d_4^{(1)}\gamma^2\bar{\alpha}^2}{d_5^{(1)}+d_4^{(1)}}\left[d_1^{(1)}\epsilon_1^{(1)}\alpha^{(1)2} + d_5^{(1)}\frac{\bar{r}^2}{\bar{\alpha}^{(1)2}} - d_5^{(1)}\right]\right\}W_n(\gamma\alpha^{(1)}\bar{\alpha}),$$

$$C_{13} = 2d_6^{(1)}(d_5^{(1)}+d_4^{(1)})\epsilon_2^{(1)}\gamma\beta^{(1)}\bar{\alpha}\,Z_{n+1}(\gamma\beta^{(1)}\bar{\alpha}) + \left\{2d_6^{(1)}(d_5^{(1)}+d_4^{(1)})n(n-1) - \right. \quad (5:37)$$

$$\left. - d_4^{(1)}\gamma^2\bar{\alpha}^2\left[d_1^{(1)}\epsilon_1^{(1)}\beta^{(1)2} + d_5^{(1)}\frac{\bar{r}^2}{\bar{\alpha}^{(1)2}} - d_5^{(1)}\right]\right\}Z_n(\gamma\beta^{(1)}\bar{\alpha}),$$

$$C_{14} = 2d_6^{(1)}(d_5^{(1)}+d_4^{(1)})\gamma\beta^{(1)}\bar{\alpha}\,W_{n+1}(\gamma\beta^{(1)}\bar{\alpha}) + \left\{2d_6^{(1)}(d_5^{(1)}+d_4^{(1)})n(n-1) - \right.$$

$$\left. - d_4^{(1)}\gamma^2\bar{\alpha}^2\left[d_1^{(1)}\epsilon_1^{(1)}\beta^{(1)2} + d_5^{(1)}\frac{\bar{r}^2}{\bar{\alpha}^{(1)2}} - d_5^{(1)}\right]\right\}W_n(\gamma\beta^{(1)}\bar{\alpha}),$$

$$C_{15} = 2n d_6^{(i)}\left[-\epsilon_3^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) + (n-1)Z_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{16} = 2n d_6^{(i)}\left[-\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) + (n-1)W_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{21} = n\zeta^{\frac{1}{2}} d_6^{(i)}\left[\epsilon_1^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) - (n-1)Z_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{22} = n\zeta^{\frac{1}{2}} d_6^{(i)}\left[\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) - (n-1)W_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{23} = n d_6^{(i)}(d_5^{(i)} + d_4^{(i)})\left[\epsilon_2^{(i)}\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}) - (n-1)Z_n(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a})\right],$$

$$C_{24} = n d_6^{(i)}(d_5^{(i)} + d_4^{(i)})\left[\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}) - (n-1)W_n(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a})\right],$$

$$C_{25} = -d_6^{(i)}\epsilon_3^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) - d_6^{(i)}\left[n(n-1) - \epsilon_3^{(i)}\zeta^2\xi^{(i)2}\bar{a}^2/2\right]Z_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}),$$

$$C_{26} = -d_6^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) - d_6^{(i)}\left[n(n-1) - \epsilon_3^{(i)}\zeta^2\xi^{(i)2}\bar{a}^2/2\right]W_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}),$$

$$C_{31} = \frac{d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}}{d_5^{(i)} + d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} - d_5^{(i)}\right]\left[-\epsilon_1^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) + nZ_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{32} = \frac{d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}}{d_5^{(i)} + d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} - d_5^{(i)}\right]\left[-\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}) + nW_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a})\right],$$

$$C_{33} = d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} - d_5^{(i)}\right]\left[-\epsilon_2^{(i)}\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}\,Z_{n+1}(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}) + nZ_n(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a})\right],$$

$$C_{34} = d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} - d_5^{(i)}\right]\left[-\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}\,W_{n+1}(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a}) + nW_n(\zeta^{\frac{1}{2}}\beta^{(i)}\bar{a})\right],$$

$$C_{35} = -n d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}\,Z_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}), \qquad \text{(5:37) (cont)}$$

$$C_{36} = -n d_4^{(i)}\zeta^{\frac{1}{2}}\bar{a}\,W_n(\zeta^{\frac{1}{2}}\xi^{(i)}\bar{a}),$$

$C_{4,i}$, $C_{5,1}$, $C_{6,i}$ $(i = 1,2,\ldots,6) = C_{3,1}$, $C_{2,i}$, $C_{1,i}$ $(i = 1,2,\ldots,6)$, respectively, with a replaced by b,

$C_{4,i}$, $C_{5,i}$, $C_{6,i}$ $(i = 7,8,\ldots,12) = -C_{3,i}$, $-C_{2,i}$, $-C_{1,i}$ $(i = 1,2,\ldots,6)$, respectively, with $\bar{a}$ and (i) replaced by $\bar{b}$ and (2),

$$C_{71} = \zeta^{\frac{1}{2}}\left[-\epsilon_1^{(i)}\zeta^{\frac{1}{2}}\xi^{(i)}b\,Z_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}b) + nZ_n(\zeta^{\frac{1}{2}}\xi^{(i)}b)\right],$$

$$C_{72} = \zeta^{\frac{1}{2}}\left[-\zeta^{\frac{1}{2}}\xi^{(i)}b\,W_{n+1}(\zeta^{\frac{1}{2}}\xi^{(i)}b) + nW_n(\zeta^{\frac{1}{2}}\xi^{(i)}b)\right],$$

$$C_{73} = (d_3^{(i)} + d_4^{(i)})\left[-\epsilon_4^{(i)}\gamma_i\beta^{(i)}b\,Z_{n+1}(\gamma_i\beta^{(i)}b) + n Z_n(\gamma_i\beta^{(i)}b)\right],$$

$$C_{74} = (d_3^{(i)} + d_4^{(i)})\left[-\gamma_i\beta^{(i)}b\,W_{n+1}(\gamma_i\beta^{(i)}b) + n W_n(\gamma_i\beta^{(i)}b)\right],$$

$$C_{75} = n Z_n(\gamma_i\xi^{(i)}b),$$

$$C_{76} = n W_n(\gamma_i\xi^{(i)}b),$$

$$C_{81} = -n\gamma_i Z_n(\gamma_i\alpha^{(i)}b),$$

$$C_{82} = -n\gamma_i W_n(\gamma_i\alpha^{(i)}b),$$

$$C_{83} = -(d_3^{(i)} + d_4^{(i)})\, n Z_n(\gamma_i\beta^{(i)}b),$$

$$C_{84} = -(d_3^{(i)} + d_4^{(i)}) n W_n(\gamma_i\beta^{(i)}b),$$

$$C_{85} = \epsilon_3^{(i)}\gamma_i\xi^{(i)}b\,Z_{n+1}(\gamma_i\xi^{(i)}b) - n Z_n(\gamma_i\xi^{(i)}b),$$

$$C_{86} = \gamma_i\xi^{(i)}b\,W_{n+1}(\gamma_i\xi^{(i)}b) - n W_n(\gamma_i\xi^{(i)}b), \qquad \text{(5:37) (cont)}$$

$$C_{91} = \frac{\gamma^2 b}{d_3^{(i)} + d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\alpha^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} + d_4^{(i)}\right] Z_n(\gamma_i\alpha^{(i)}b),$$

$$C_{92} = \frac{\gamma^2 b}{d_3^{(i)} + d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\alpha^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} + d_4^{(i)}\right] W_n(\gamma_i\alpha^{(i)}b),$$

$$C_{93} = \gamma_i b\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} + d_4^{(i)}\right] Z_n(\gamma_i\beta^{(i)}b),$$

$$C_{94} = \gamma_i b\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{Z^2}{\xi^{(i)2}} + d_4^{(i)}\right] W_n(\gamma_i\beta^{(i)}b),$$

$$C_{95} = C_{96} = 0,$$

$C_{7,i}$, $C_{8,i}$, $C_{9,i}$ $(i = 7,8,\ldots,12) = -C_{7,i}$, $-C_{8,i}$, $-C_{9,i}$ $(i = 1,2,\ldots,6)$, respectively, with (i) replaced by (2),

$C_{10,i}$, $C_{11,i}$, $C_{12,i}$ $(i = 7,8,\ldots,12) = -C_{9,i}$, $-C_{8,i}$, $-C_{7,i}$ $(i = 1,2,\ldots,6)$, respectively, with b and (i) replaced by c and (2),

$C_{10,i}$, $C_{11,i}$, $C_{12,i}$ $(i = 13,14, \ldots ,18) = C_{9,i}$, $C_{8,i}$, $C_{7,i}$ $(i = 1,2, \ldots ,6)$, respectively, with $\bar{b}$ and ω replaced by $\bar{c}$ and ω,

$C_{13,i}$, $C_{14,i}$, $C_{15,i}$ $(i = 7,8, \ldots ,12) = -C_{1,i}$, $-C_{2,i}$, $-C_{3,i}$ $(i = 1,2, \ldots ,6)$, respectively, with $\bar{a}$ and ω replaced by $\bar{c}$ and ω,

$C_{13,i}$, $C_{14,i}$, $C_{15,i}$ $(i = 13,14, \ldots ,18) = C_{1,i}$, $C_{2,i}$, $C_{3,i}$ $(i = 1,2, \ldots ,6)$, respectively, with $\bar{a}$ and ω replaced by $\bar{c}$ and ω,

$C_{16,i}$, $C_{17,i}$, $C_{18,i}$ $(i = 13,14, \ldots ,18) = C_{3,i}$, $C_{2,i}$, $C_{1,i}$ $(i = 1,2, \ldots ,6)$, respectively, with $\bar{a}$ and ω replaced by $\bar{d}$ and ω.

(5:37)
(cont)

3. The Ω-ξ Plane

The plane of coordinates defined by the nondimensional frequency Ω as ordinate and the nondimensional wave number ξ as abscissa will be referred to interchangeably as the Ω-ξ plane or the frequency spectrum. Equation (5:34), being a transcendental relationship between frequency, wave number, and number of circumferential waves n, yields an infinite family of values of Ω for each chosen value of ξ and n. If the values chosen for ξ are dense enough, there results, rather than discreet Ω-ξ point sets, an infinite family of curves for each choice of n. Each of these curves is a branch of the frequency equation. Each branch of the frequency equation originates at a specific value of Ω (called the cut-off frequency) when ξ is zero and, as ξ increases, passes across quadrant 1 of the Ω-ξ plane (see Figure 6). The branches are frequently given names which are indicative of the

character of the displacement field at the cut-off frequency of the branch (for example, flexure mode, breathing mode, etc.). Although the displacement field will vary markedly as γ is increased along a particular branch, the cut-off frequency names are retained for purposes of identification and convenience.

The coordinates Ω and γ have been chosen to depict the behavior of the frequency equation because of the volume of information revealed by the branch curves when plotted against this choice of variables. Not only are the values of the frequencies for each value of γ apparent at a glance, but since

$$\frac{\Omega}{\gamma} = \frac{V}{V_{s^{(1)}}}, \tag{5:38}$$

it is apparent that the phase velocity of a traveling wave of given wavelength and frequency is available from the spectrum as the slope of a line from the origin through the point with coordinates Ω and γ. Furthermore, since the group velocity (the velocity with which energy is transported) is obtained as the derivative of the frequency with respect to the wave number, it follows that

$$\frac{d\omega}{d k} = V_g = V_{s^{(1)}} \frac{d\Omega}{d\gamma},$$

or

$$\frac{d\Omega}{d\gamma} = \frac{V_g}{V_{s^{(1)}}}. \tag{5:39}$$

Hence the slope at any point on a branch of the frequency equation is the nondimensionalized group velocity of a wave with the frequency and wave number given by these coordinates.

Historically the branches of the frequency equation were plotted in coordinates of phase velocity versus wave number. Since, at the cut-off frequencies, the phase velocity is infinite, such plots could not reveal the value of the cut-off frequencies. The frequency spectrum circumvents this problem.

From the point of view of analytical (as opposed to numerical) analysis, there are other advantages to the use of the Ω-ζ coordinates for analysis of the frequency equation. Chapter IV revealed that the turning point numbers, the commencement, and the termination of complex domains of the radial wave numbers (for real ζ) are all readily identified in terms of values of the ratio Ω/ζ. Thus, on the frequency spectrum the turning point numbers, and complex zone numbers represent rays of specific slope extending from the origin of coordinates across quadrant 1 of the Ω-ζ plane. There are three turning point lines and two complex zone lines (for more anisotropic materials) for each layer of the composite cylinder. Any analysis must necessarily be concerned with the ordering of the slopes of this array of lines since the nature of the elements comprising the frequency equation determinant vary as Ω/ζ ranges in its values from greater than to less than their slopes.

Since the different degrees of transverse isotropy under consideration may or may not produce complex radial wave numbers, but always admit the turning point values, the ordering of the turning point lines will be considered initially. The turning point line for $g^{(1)}$ has the same slope as that of $p^{(1)}$

(of the same layer) or the lowest of $q^{(i)}$ if $p^{(i)}$ does not vanish, consequently, there are at best six distinct rays emanating from the origin of the frequency spectrum which will be turning point lines. These rays divide the frequency spectrum into seven sectors (Figure 6), the lowest being numbered 1 and the others numbered counter-clockwise. The slopes of turning point lines, which form the sector boundaries, are the six numbers $b^{(i)}, \gamma^{(i)}$ $(i = 1,2,3)$. Moreover, for a given layer $\gamma^{(i)} > b^{(i)}$ since $c_{33}^{(i)} > c_{44}^{(i)}$. Since this is the only restriction in the ordering of the turning point lines, there are 90 distinct ways that the seven sectors can be bounded. Many of these 90 orderings have little other than pedagogical significance.

The field of 90 orderings of the sector lines is narrowed by the following observations. Light weight sandwich constructions utilize low strength, low density cores which carry transverse shear in their primary rôle of separation and stabilization [74] of the outer layers. The outer layers, or facings, on the other hand, are high strength, more dense materials. Chu [42,75] and Yu [41,76,77] have used such characteristics for the three layers in their approximate theories for light-weight aniaotropic and isotropic sandwich shells. Since $c_{33}^{(i)}$ and $c_{44}^{(i)}$ are directly related to the axial Young's modulus and shear modulus, respectively, and are also proportional to the velocity ratios $\gamma^{(i)}$ and $b^{(i)}$, light weight sandwich shells will be specified to be those shells for which

$$\eta_b^{(2)} < \eta_b^{(1)}, \quad (i = 1,3). \tag{5:40}$$

This requirement can be expanded into the following six orderings of the turning point line slopes, none of which restrict the theory to the requirement for identical facings

$$b^{(2)} < b^{(3)} < \begin{cases} b^{(3)} < \eta_b^{(3)} < b^{(1)} < \eta_b^{(1)} \\ b^{(1)} < b^{(3)} < \eta_b^{(3)} < \eta_b^{(1)} \\ b^{(1)} < b^{(3)} < \eta_b^{(3)} < \eta_b^{(1)} \end{cases} \quad \text{(a)}$$

and light weight sandwich shells (5:41)

$$b^{(2)} < \eta_b^{(2)} < \begin{cases} b^{(3)} < \eta_b^{(1)} < b^{(3)} < \eta_b^{(3)} \\ b^{(1)} < b^{(3)} < \eta_b^{(3)} < \eta_b^{(3)} \\ b^{(3)} < b^{(1)} < \eta_b^{(3)} < \eta_b^{(3)}. \end{cases} \quad \text{(b)}$$

If, on the other hand, the core layer of the cylinder is a high strength, high density material relative to the facing layers, then the axial Young's modulus and shear modulus of the core will be larger than the corresponding moduli of the layers. Such core layered shells will be restricted by the following ordering of the velocity ratios

$$\left.\begin{matrix} b^{(3)} < \eta_b^{(3)} < b^{(1)} < \eta_b^{(1)} \\ b^{(3)} < b^{(1)} < \eta_b^{(3)} < \eta_b^{(1)} \\ b^{(3)} < b^{(3)} < \eta_b^{(3)} < \eta_b^{(1)} \end{matrix}\right\} < b^{(2)} < \eta_b^{(2)}, \quad \text{(a)}$$

and core layered shells (5:42)

$$\left.\begin{matrix} b^{(1)} < \eta_b^{(1)} < b^{(3)} < \eta_b^{(3)} \\ b^{(1)} < b^{(3)} < \eta_b^{(1)} < \eta_b^{(3)} \\ b^{(3)} < b^{(1)} < \eta_b^{(1)} < \eta_b^{(3)} \end{matrix}\right\} < b^{(2)} < \eta_b^{(2)}. \quad \text{(b)}$$

The above two listings of turning point line slopes do not include the important class of shells for which the strength of the individual layers is comparable. Again six

possibilities will be listed - based on the restriction that the facing layers be of the same material. Thus

$$\beta^{(1)} = \beta^{(3)}, \quad \eta^{(1)} = \eta^{(3)},$$

and

$$\begin{aligned} &\beta^{(2)} < \eta^{(2)} < \beta^{(1)} < \eta^{(1)}, \\ &\beta^{(2)} < \beta^{(1)} < \eta^{(2)} < \eta^{(1)}, \qquad \text{(a)} \\ &\beta^{(1)} < \beta^{(2)} < \eta^{(1)} < \eta^{(2)}, \end{aligned}$$

and general shells (5:43)

$$\begin{aligned} &\beta^{(2)} < \eta^{(2)} < \beta^{(1)} < \eta^{(1)}, \\ &\beta^{(2)} < \beta^{(1)} < \eta^{(2)} < \eta^{(1)}, \qquad \text{(b)} \\ &\beta^{(1)} < \beta^{(2)} < \eta^{(1)} < \eta^{(2)}. \end{aligned}$$

Inequalities (5:43) actually bridge the gap between (5:41) and (5:42), that is, the first of (5:43a) corresponds to all of (5:41) and the first of (5:43b) corresponds to all of (5:42) for the case of identical facing materials. Similarly, if all three materials are the same, the last of (5:43a and b) correspond to the single requirement for the turning point line slopes of a single layered cylinder if the first and third inequality signs are changed to equality signs. All eighteen orderings listed above in no manner affect the geometry of the shell layers, but do represent a cross section of useful possibilities of layer strength and density ratios which, if properly pursued, will reveal the trend behavior of the 90 possible cases.

Turning now to the slopes, $Z_{L1}^{(i)}$ and $Z_{L3}^{(i)}$, it was established in Chapter IV that (a) only for more anisotropic materials will these lines exist, and (b) the slopes of these

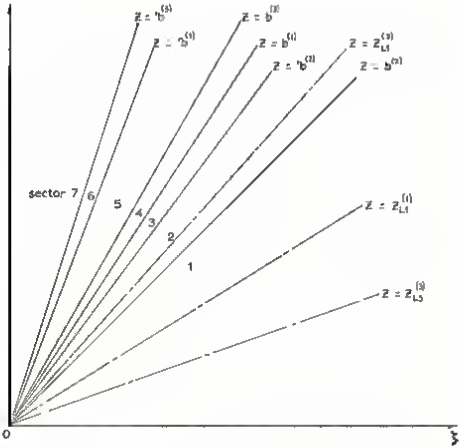

FIGURE 6. Ω-ζ Plane with Zone Lines

The Ω-ζ plane for a light weight sandwich shell depicted by the velocity ratio ordering

$$b^{(6)} < b^{(5)} < b^{(4)} < b^{(2)} < b^{(7)} < b^{(1)} < b^{(3)}$$

of equation (5:41b). The outer layer is assumed to be class I more anisotropic, with $b^{(4)} < Z_{L1}^{(3)} < b^{(3)}$ and $Z_{L3}^{(3)} < b^{(4)}$. The inner layer is assumed to be class II more anisotropic, with $Z_{L1}^{(1)} < b^{(1)}$ and $Z_{L3}^{(1)} < 0$. The middle layer is less anisotropic. The broken lines are complex zone lines.

lines are always less than the smallest turning point line slope for a given material. Restrictions, relative to $c_{11}^{(i)}$ and the three constants $c_{13}^{(i)}$, $c_{33}^{(i)}$, and $c_{44}^{(i)}$ were listed in Chapter IV by which it can be ascertained whether the lowest complex zone line has zero or positive slope. Consideration of these facts in light of inequalities (5:41), (5:42), and (5:43) indicates that a complex zone for an outer layer of a light weight sandwich shell may have its upper boundary line in any sector from 1 thru 5, the same being true for the core layer of the shell class described in (5:42). Within the framework of (5:41), (5:42), and (5:43) there is no more specific way to indicate the occurrence of the complex zones of $\underline{q}^{(i)}$ and $\underline{p}^{(i)}$.

The above comments indicate the need for the proper representation of the C_{ij} when Ω/γ has a turning point line, or complex zone line value. In the next two sections of this chapter these formulations will be obtained.

4. Turning Point Line Expressions for the C_{ij}

As mentioned above, there are, in general, 6 distinct values of Ω/γ which are turning point numbers. Only the wave number $q^{(i)}$ vanishes for the turning point number $v^{(i)}$ which is the largest of the two for a specific layer, and for class I or II anisotropies, respectively, $g^{(i)}$ and $p^{(i)}$ or $g^{(i)}$ and $q^{(i)}$ vanish for the lower value, $b^{(i)}$. Consequently, recalling the construction of the frequency equation determinant, it follows that the C_{ij} are affected in groups of two or four columns

when Z attains a turning point value. Thus, any given turning point number only affects a minor portion of the formulation of equation (5:37).

The proper form of the C_{ij} can be obtained by use of the appropriate solution for $\varphi^{(i)}$ or $f^{(i)}$ given in Chapter IV, for example, if $g^{(i)} = 0$ equation (4:8) can be used in the appropriate portions of equations (5:33) to rederive the C_{ij} for columns 17 and 18 in (5:34). This method is tedious since the C_{ij} found by the above process will be unpredictably different depending on whether n is, or is not, zero.

The awkwardness of this situation is eliminated by utilization of equations (4:10), which indicate the alteration in the form of the integration constants as a radial wave number vanishes. These equations can be used in applying the limit to the C_{ij} of equations (5:37) to obtain the proper turning point line formulations. Reference to (4:10a and b), for $n \neq 0$, shows that no order of differentiation effects the form of the constants A(0) or B(0). However, when $n = 0$, the fact that the limit of a derivative is not necessarily the same as the derivative of a limit must be entertained. Equations (5:33) show that either $\varphi^{(i)}$ or $f^{(i)}$ are differentiated at best only once in obtaining the C_{ij}. Hence, for $n = 0$, the Bessel functions of order 1 in equations (5:37) derive from zero order Bessel functions. Thus, taking the limit, as the radial wave number vanishes, of a W_1 Beseel function must be treated as taking the limit of the derivative of (4:10c) rather than that of (4:10b) with $n = 1$. The difference in

easy to illustrate. Thus, from (4:10b) with n = 1

$$\mathfrak{H}_3(k\varepsilon)\frac{2}{\pi k\varepsilon} \rightarrow \mathfrak{H}_3(0),$$

and from (4:10c), taking the derivative, and then the limit

$$\mathfrak{H}_3(k\varepsilon)\frac{2}{\pi} \rightarrow \mathfrak{H}_3(0). \qquad (5:44)$$

Equation (5:44) is the proper limit for the W_1 Bessel functions in (5:35) for the axisymmetric case n = 0.

In order to utilize (5:44) and (4:10) in (5:35) the appropriate asymptotic expressions for small arguments are introduced in place of the Bessel functions. The constant groups, for instance 2/π as in (5:44), are factored out of the terms in accordance with the limits of (4:10) and (5:44) and are assumed as part of the integration constants and hence not part of the C_{ij}.

The altered formulations of the C_{ij} may be written by inspection as

$$C_{11} = \gamma\left[2d_6^{(0)}n(n-1) - \frac{d_4^{(0)}d_5^{(0)}}{d_5^{(0)}+d_4^{(0)}}\gamma^2\bar{\alpha}^2\left(\frac{\bar{z}^2}{\bar{y}^2}-1\right)\right]\bar{\alpha}^n,$$

$$C_{12} = \begin{cases} \gamma\left[2d_6^{(0)}n(n+1) - \dfrac{d_4^{(0)}d_5^{(0)}}{d_5^{(0)}+d_4^{(0)}}\gamma^2\bar{\alpha}^2\left(\dfrac{\bar{z}^2}{\bar{y}^2}-1\right)\right]\bar{\alpha}^{-n}, \\ -2d_6^{(0)}\gamma - \dfrac{d_4^{(0)}d_5^{(0)}}{d_5^{(0)}+d_4^{(0)}}\gamma^3\bar{\alpha}^2\left(\dfrac{\bar{z}^2}{\bar{y}^2}-1\right)\ln\bar{\alpha}\ , \quad n=0, \end{cases}$$

$$C_{13} = \left[2d_6^{(0)}(d_5^{(0)}+d_4^{(0)})n(n-1) - d_4^{(0)}d_5^{(0)}\gamma^2\bar{\alpha}^2\left(\frac{\bar{z}^2}{\bar{y}^2}-1\right)\right]\bar{\alpha}^n, \qquad (5:45)$$

$$C_{14} = \begin{cases} \left[2d_6^{(0)}(d_5^{(0)}+d_4^{(0)})n(n+1) - d_4^{(0)}d_5^{(0)}\gamma^2\bar{\alpha}^2\left(\dfrac{\bar{z}^2}{\bar{y}^2}-1\right)\right]\bar{\alpha}^{-n}, \\ -2d_6^{(0)}(d_5^{(0)}+d_4^{(0)}) - d_4^{(0)}d_5^{(0)}\gamma^2\bar{\alpha}^2\left(\dfrac{\bar{z}^2}{\bar{y}^2}-1\right)\ln\bar{\alpha}\ , \quad n=0, \end{cases}$$

$$C_{15} = 2d_6^{(0)}n(n-1)\bar{\alpha}^n,$$

$$C_{16} = \begin{cases} -2d_6^{(i)} n(n+1)\bar{\alpha}^{-n} \\ 0\ ,\ n=0, \end{cases}$$

$$C_{21} = -d_6^{(i)} \xi n(n-1)\bar{\alpha}^{n},$$

$$C_{22} = \begin{cases} d_6^{(i)} \xi n(n+1)\bar{\alpha}^{-n} \\ 0\ ,\ n=0, \end{cases}$$

$$C_{23} = d_5^{(i)}(d_5^{(i)} + d_4^{(i)}) n(n-1)\bar{\alpha}^{n},$$

$$C_{24} = \begin{cases} d_6^{(i)}(d_5^{(i)} + d_4^{(i)}) n(n+1)\bar{\alpha}^{-n} \\ 0,\ n=0, \end{cases}$$

$$C_{25} = -d_6^{(i)} n(n-1)\bar{\alpha}^{n},$$

$$C_{26} = \begin{cases} -d_6^{(i)} n(n+1)\bar{\alpha}^{-n} \\ d_6^{(i)},\ n=0, \end{cases}$$

$$C_{31} = \frac{-d_4^{(i)}\xi^2\bar{\alpha}}{d_5^{(i)}+d_4^{(i)}}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right) n\bar{\alpha}^{n},$$

$$C_{32} = \begin{cases} \frac{d_4^{(i)}\xi^2\bar{\alpha}}{d_5^{(i)}+d_4^{(i)}}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right) n\bar{\alpha}^{-n} \\ \frac{-d_4^{(i)}\xi^2\bar{\alpha}}{d_5^{(i)}+d_4^{(i)}}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right),\ n=0, \end{cases} \qquad \begin{matrix}(5:45)\\(\text{cont}\end{matrix}$$

$$C_{33} = -\xi\bar{\alpha}\, d_4^{(i)}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right) n\bar{\alpha}^{n},$$

$$C_{34} = \begin{cases} \xi\bar{\alpha}\, d_4^{(i)}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right) n\bar{\alpha}^{-n} \\ -\bar{\alpha}\, d_4^{(i)}\left(d_5^{(i)} + d_4^{(i)}\frac{\xi^2}{\bar{\alpha}^{(i)2}}\right),\ n=0, \end{cases}$$

$$C_{35} = -n\xi\bar{\alpha}\, d_4^{(i)}\bar{\alpha}^{n},$$

$$C_{36} = \begin{cases} -n\xi\bar{\alpha}\, d_4^{(i)}\bar{\alpha}^{-n} \\ 0\ ,\ n=0, \end{cases}$$

$$C_{71} = n\xi^{n},$$

$$C_{72} = \begin{cases} n\xi^{-n} \\ \xi\ ,\ n=0, \end{cases}$$

$$C_{73} = (d_5^{(0)} + d_4^{(0)}) n b^n,$$

$$C_{74} = \begin{cases} -(d_5^{(0)} + d_4^{(0)}) n b^{-n} \\ (d_5^{(0)} + d_4^{(0)}), \ n=0, \end{cases}$$

$$C_{75} = n b^n,$$

$$C_{76} = \begin{cases} n b^{-n} \\ 0 \ , \ n=0, \end{cases}$$

$$C_{81} = -\frac{1}{2} n b^n,$$

$$C_{82} = \begin{cases} \frac{1}{2} n b^{-n} \\ 0, \ n=0, \end{cases}$$

$$C_{83} = -(d_5^{(0)} + d_4^{(0)}) n b^n,$$

$$C_{84} = \begin{cases} -(d_5^{(0)} + d_4^{(0)}) n b^{-n} \\ 0, n=0, \end{cases}$$

$$C_{85} = -n b^n,$$

(5:45) (cont.)

$$C_{86} = \begin{cases} n b^{-n} \\ -1 \ , \ n=0, \end{cases}$$

$$C_{91} = \frac{d_4^{(0)} \gamma^2 b}{d_5^{(0)} + d_4^{(0)}} \left(1 - \frac{\bar{z}^2}{b^{(0)}}\right) b^n,$$

$$C_{92} = \begin{cases} \frac{d_4^{(0)} \gamma^2 b}{d_5^{(0)} + d_4^{(0)}} \left(1 - \frac{\bar{z}^2}{b^{(0)}}\right) b^{-n} \\ \frac{d_4^{(1)} \gamma^2 b}{d_5^{(1)} + d_4^{(1)}} \left(1 - \frac{\bar{z}^2}{b^{(1)}}\right) \ln b \ , \ n=0, \end{cases}$$

$$C_{93} = \gamma^2 b \, d_4^{(0)} \left(1 - \frac{\bar{z}^2}{b^{(0)}}\right) b^n,$$

$$C_{94} = \begin{cases} \gamma^2 b \, d_4^{(0)} \left(1 - \frac{\bar{z}^2}{b^{(0)}}\right) b^{-n} \\ \gamma^2 b \, d_4^{(1)} \left(1 - \frac{\bar{z}^2}{b^{(1)}}\right) \ln b \ , \ n=0 \end{cases}$$

$$C_{95} = C_{96} = 0$$

The remaining C_{ij} are formed from the above by use of the instructions included in equations (5:37).

Equations (5:45) are not to be used as a listing of the general elements of the frequency equation as are equations (5:37). Only columns from the frequency equation for which the radial wave number is zero at a specific choice of Z should be composed from the corresponding column elements listed in equation (5:45).

5. Complex Zone Line Expressions for the C_{ij}

The complex zones of the radial wave numbers occur only in $p^{(i)}$ and $q^{(i)}$, hence columns (6i - 1) and (6i) (i = 1,2,3) of the frequency equation determinant do not involve Bessel functions of complex argument. Equation (4:45) suffices as the solution for $\varphi^{(i)}$ when $p^{(i)} = q^{(i)}$. This solution may be written as

$$\varphi^{(i)}(r) = \frac{\Sigma}{\frac{(i)}{\delta_2^2 + \delta_3^2} \frac{\pi}{k^{(i)} c_{44}^{(i)}}} \varphi_1^{(i)}(r) + \widetilde{\varphi}_2^{(i)}(r),$$

where $\varphi_1^{(i)}$ is the same as used previously in the general solution but

$$\widetilde{\varphi}_2^{(i)}(r) = \overline{r}\left(\widetilde{A}_2^{(i)} Z_{nn}(\xi e^{(i)} r) + \widetilde{B}_2^{(i)} W_{nn}(\xi e^{(i)} r)\right). \qquad (5:46)$$

Consequently, only columns (6i - 3) and (6i - 2) (i = 1,2,3) are affected by the occurrence of complex zone line values for Z. The new forms for the altered C_{ij} will be listed below. It is important to recall, in using these terms, that since the three layers are generally dissimilar, only two columns of the frequency equation determinant will be affected for any given complex zone line value of Z.

$$C_{13} = 2\,s_6^{(i)}(s_5^{(i)} + s_4^{(i)})\epsilon_1^{(i)}\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}^2\,Z_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) + \bar{\alpha}\Big\{2\,s_6^{(i)}(s_5^{(i)} + s_4^{(i)})(n+1)(n-2) - \\ - s_4^{(i)}\underline{\gamma}^2\bar{\alpha}^2\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} + s_5^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} - s_5^{(i)}\Big]\Big\}Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}),$$

$$C_{14} = 2\,s_6^{(i)}(s_5^{(i)} + s_4^{(i)})\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}^2\,W_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) + \bar{\alpha}\Big\{2\,s_6^{(i)}(s_5^{(i)} + s_4^{(i)})(n+1)(n-2) - \\ - s_4^{(i)}\underline{\gamma}^2\bar{\alpha}^2\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} + s_5^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} - s_6^{(i)}\Big]\Big\}W_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}),$$

$$C_{23} = 2\,s_6^{(i)}\bar{\alpha}\,n(s_5^{(i)} + s_4^{(i)})\Big[\epsilon_1^{(i)}\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}\,Z_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) - (n+1)Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha})\Big],$$

$$C_{24} = 2\,s_6^{(i)}\bar{\alpha}\,n(s_5^{(i)} + s_4^{(i)})\Big[\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}\,W_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) - (n+1)W_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha})\Big],$$

$$C_{33} = s_4^{(i)}\bar{\alpha}^2\underline{\gamma}\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} - s_4^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} - s_5^{(i)}\Big]\Big[-\epsilon_1^{(i)}\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}\,Z_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) + (n+2)Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha})\Big],$$

$$C_{34} = s_4^{(i)}\bar{\alpha}^2\underline{\gamma}\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} - s_4^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} - s_5^{(i)}\Big]\Big[-\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}\,W_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha}) + (n+2)Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\bar{\alpha})\Big],$$

$$C_{43} = (s_5^{(i)} + s_4^{(i)})\underline{b}\Big[-\epsilon_1^{(i)}\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}\,Z_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}) + (n+2)Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b})\Big], \qquad (5:47)$$

$$C_{44} = (s_5^{(i)} + s_4^{(i)})\underline{b}\Big[-\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}\,W_{n+2}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}) + (n+2)W_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b})\Big],$$

$$C_{83} = -n\underline{b}\,(s_5^{(i)} + s_4^{(i)})\,Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}),$$

$$C_{84} = n\underline{b}\,(s_5^{(i)} + s_4^{(i)})W_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}),$$

$$C_{93} = \underline{\gamma}\underline{b}\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} - s_4^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} + s_4^{(i)}\Big]Z_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}),$$

$$C_{94} = \underline{\gamma}\underline{b}\Big[s_1^{(i)}\epsilon_1^{(i)}\underline{\alpha}^{(i)2} - s_4^{(i)}\frac{\bar{Z}^2}{\underline{\alpha}^{(i)2}} + s_4^{(i)}\Big]W_{n+1}(\underline{\gamma}\,\underline{\alpha}^{(i)}\underline{b}).$$

Elements of columns 8, 9, 15, and 16 are formed from the above by use of the instructions in equation (5:37).

6. The Frequency Equation Representation in Complex Zones

For more isotropic materials it was shown in Chapter IV that $\underline{p}^{(i)}$ and $\underline{q}^{(i)}$ are complex conjugate numbers when $Z_{tt}^{(i)} > Z > Z_{tt}^{(i)}$. Moreover, $\underline{p}^{(i)}$ and $\underline{q}^{(i)}$ are negative complex conjugate numbers.

Denoting the real part of a complex number by R and the imaginary part by I, the following formulas result for use in complex zones of the radial wave numbers. Since the equations are general, superscripts denoting material layer association are for the moment disregarded.

$$
\begin{aligned}
&\underline{\alpha}^2 = \underline{\gamma}^2(R\alpha^2 + iI\alpha^2),\\
&\underline{\beta}^2 = \underline{\gamma}^2(R\beta^2 - iI\beta^2),\\
&\underline{\rho} = \underline{\gamma}(R\alpha + iI\alpha),\\
&\underline{\sigma} = -\underline{\gamma}(R\alpha - iI\alpha),\\
&J_n(\underline{\rho}\bar{r}) = RJ_n(\underline{\gamma}\alpha\bar{r}) + iIJ_n(\underline{\gamma}\alpha\bar{r}),\\
&Y_n(\underline{\rho}\bar{r}) = RY_n(\underline{\gamma}\alpha\bar{r}) + iIY_n(\underline{\gamma}\alpha\bar{r}),\\
&J_n(\underline{\sigma}\bar{r}) = (-1)^n J_n^*(\underline{\rho}\bar{r}),\\
&Y_n(\underline{\sigma}\bar{r}) = (-1)^n Y_n^*(\underline{\rho}\bar{r}),
\end{aligned}
\qquad (5:48)
$$

where the asterisk denotes complex conjugate.

The following terms from (5:37) are noted for convenience:

$$
\begin{aligned}
&d_1\alpha^2 + d_3\frac{\bar{z}^2}{h^2} - d_5 = d_1R\alpha^2 + d_3\frac{\bar{z}^2}{h^2} - d_5 + id_1I\alpha^2 = RF_1 + iIF_1,\\
&d_1\beta^2 + d_3\frac{\bar{z}^2}{h^2} - d_5 = RF_1 - iIF_1,\\
&d_1\alpha^2 - d_4\frac{\bar{z}^2}{h^2} - d_4 = d_1R\alpha^2 - d_4\frac{\bar{z}^2}{h^2} - d_4 + id_1I\alpha^2 = RF_2 + iIF_2,\\
&d_1\beta^2 - d_4\frac{\bar{z}^2}{h^2} - d_4 = RF_2 - iIF_2,\\
&d_1\alpha^2 - d_4\frac{\bar{z}^2}{h^2} - d_5 = d_1R\alpha^2 - d_4\frac{\bar{z}^2}{h^2} - d_5 + id_1I\alpha^2 = RF_3 + iIF_3,\\
&d_1\beta^2 - d_4\frac{\bar{z}^2}{h^2} - d_5 = RF_3 - iIF_3.
\end{aligned}
\qquad (5:49)
$$

The frequency equation is affected in columns (6i - 5),

(6i - 4), (6i - 3), and (6i - 2) (i = 1,2,3) when $p^{(i)}$ and $q^{(i)}$ are complex. Study of the element listing in (5:37) shows that, element by element, column (6i - n) and (6i - n + 2) (i = 1,2,3; n = 4,5) are complex conjugate pairs with every element in the $q^{(i)}$ columns modified by the same factor $(d_5^{(i)} + d_4^{(i)})/\gamma$ relative to the $p^{(i)}$ columns. Assuming that i = 1, the frequency equation may be written as

$$\det|C_{ij}| = \det|C_{i1}C_{i2}C_{i3}C_{i4}\mathcal{B}| = 0,$$

where $\mathcal{B}$ is the last 14 columns of the frequency equation, and C_{ij} (j = 1,2,3,4) are the four columns which are complex. The C_{ij} can be written

$$C_{i1} = C_{i1} + iC_{i3}\,, \qquad C_{i3} = C_{i1} - iC_{i3}\,,$$
$$C_{i2} = C_{i2} + iC_{i4}\,, \qquad C_{i4} = C_{i2} - iC_{i4}\,. \tag{5:50}$$

Substituting these expressions into $\det|C_{ij}|$ and expanding according to the addition rules for determinants yields

$$\det|C_{ij}| = -4\det|C_{i1}C_{i2}C_{i3}C_{i4}\mathcal{B}| = 0 \tag{5:51}$$

Thus, when a given set of $p^{(i)}$ and $q^{(i)}$ are complex, the 4 columns affected by the transition can be replaced according to (5:50) and (5:51) by the real and imaginary parts of the column elements of which $p^{(i)}$ is the radial wave number. The same substitution could as well be made in terms of the $q^{(i)}$ column elements. Using (5:50) and (5:37) the elements of the frequency equation which are affected by complex $p^{(i)}$ and $q^{(i)}$ are given as

$$C_{11} = \left\{ 2d_6^{(1)} \gamma^2 \bar{\alpha} \left[R\mathfrak{B}^{(1)} R J_{n+1} - I\mathfrak{B}^{(1)} I J_{n+1} \right] + \gamma \left[2d_6^{(1)} n(n-1) - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^2 \bar{\alpha}^2 R F_1^{(1)} \right] R J_n + \right.$$

$$\left. + \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^3 \bar{\alpha}^2 I F_1^{(1)} I J_n \right\} +$$

$$+ i \left\{ \left[2d_6^{(1)} \gamma^2 \bar{\alpha} \left[R\mathfrak{B}^{(1)} I J_{n+1} + I\mathfrak{B}^{(1)} R J_{n+1} \right] + \gamma \left[2d_6^{(1)} n(n-1) - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^2 \bar{\alpha}^2 R F_1^{(1)} \right] I J_n - \right.$$

$$\left. - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^3 \bar{\alpha}^2 I F_1^{(1)} R J_n \right\} ,$$

$$C_{12} = \left\{ 2d_6^{(1)} \gamma^2 \bar{\alpha} \left[R\mathfrak{B}^{(1)} R Y_{n+1} - I\mathfrak{B}^{(1)} I Y_{n+1} \right] + \gamma \left[2d_6^{(1)} n(n-1) - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^2 \bar{\alpha}^2 R F_1^{(1)} \right] R Y_n + \right.$$

$$\left. + \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^3 \bar{\alpha}^2 I F_1^{(1)} I Y_n \right\} +$$

$$+ i \left\{ \left[2d_6^{(1)} \gamma^2 \bar{\alpha} \left[R\mathfrak{B}^{(1)} I Y_{n+1} + I\mathfrak{B}^{(1)} R Y_{n+1} \right] + \gamma \left[2d_6^{(1)} n(n-1) - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^2 \bar{\alpha}^2 R F_1^{(1)} \right] I Y_n - \right.$$

$$\left. - \frac{d_4^{(1)}}{d_5^{(1)} + d_6^{(1)}} \gamma^3 \bar{\alpha}^2 I F_1^{(1)} R Y_n \right\} ,$$

$$C_{21} = d_6^{(1)} \gamma n \left\{ \gamma \bar{\alpha} \left[R\mathfrak{B}^{(1)} R J_{n+1} - I\mathfrak{B}^{(1)} I J_{n+1} \right] - (n-1) R J_n \right\} +$$

$$+ i d_6^{(1)} \gamma n \left\{ \gamma \bar{\alpha} \left[R\mathfrak{B}^{(1)} I J_{n+1} + I\mathfrak{B}^{(1)} R J_{n+1} \right] - (n-1) I J_n \right\} ; \qquad (5:52)$$

$$C_{22} = d_6^{(1)} \gamma n \left\{ \gamma \bar{\alpha} \left[R\mathfrak{B}^{(1)} R Y_{n+1} - I\mathfrak{B}^{(1)} I J_{n+1} \right] - (n-1) R J_n \right\} +$$

$$+ i d_6^{(1)} \gamma n \left\{ \gamma \bar{\alpha} \left[R\mathfrak{B}^{(1)} I Y_{n+1} + I\mathfrak{B}^{(1)} R Y_{n+1} \right] - (n-1) I Y_n \right\} ;$$

$$C_{31} = \frac{d_4^{(1)} \gamma^2 \bar{\alpha}}{d_5^{(1)} + d_6^{(1)}} \left\{ R F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} R J_{n+1} - I\mathfrak{B}^{(1)} I J_{n+1} \right) + n R J_n \right] - \right.$$

$$\left. - I F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} I J_{n+1} + I\mathfrak{B}^{(1)} R J_{n+1} \right) + n I J_n \right] \right\} +$$

$$+ i \frac{d_4^{(1)} \gamma^2 \bar{\alpha}}{d_5^{(1)} + d_6^{(1)}} \left\{ I F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} R J_{n+1} - I\mathfrak{B}^{(1)} I J_{n+1} \right) + n R J_n \right] + \right.$$

$$\left. + R F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} I J_{n+1} + I\mathfrak{B}^{(1)} R J_{n+1} \right) + n I J_n \right] \right\} ,$$

$$C_{32} = \frac{d_6^{(1)} \gamma^2 \bar{\alpha}}{d_5^{(1)} + d_6^{(1)}} \left\{ R F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} R Y_{n+1} - I\mathfrak{B}^{(1)} I Y_{n+1} \right) + n R Y_n \right] - \right.$$

$$\left. - I F_3^{(1)} \left[-\gamma \bar{\alpha} \left(R\mathfrak{B}^{(1)} I Y_{n+1} + I\mathfrak{B}^{(1)} R Y_{n+1} \right) + n I Y_n \right] \right\} + (\text{cont})$$

$$+(\text{cont})\ i\frac{d_5^{(i)}\gamma^2\bar{a}}{d_5^{(i)}+d_4^{(i)}}\left\{\left[F_3^{(i)}\left[-\gamma\bar{a}\left(R\alpha^{(i)}RY_{n+1}-I\alpha^{(i)}IY_{n+1}\right)+nRY_n\right]+\right.\right.$$

$$\left.\left.+RF_3^{(i)}\left[-\gamma\bar{a}\left(R\alpha^{(i)}IY_{n+1}+I\alpha^{(i)}RY_{n+1}\right)+nIY_n\right]\right\},$$

$$C_{71}=\gamma\left\{\left[-\gamma\bar{b}\left[R\alpha^{(i)}RJ_{n+1}-I\alpha^{(i)}IJ_{n+1}\right]+nRJ_n\right]\right\}+$$

$$+i\gamma\left\{\left[-\gamma\bar{b}\left[R\alpha^{(i)}IJ_{n+1}+I\alpha^{(i)}RJ_{n+1}\right]+nIJ_n\right]\right\},$$

$$C_{72}=\gamma\left\{\left[-\gamma\bar{b}\left[R\alpha^{(i)}RY_{n+1}-I\alpha^{(i)}IY_{n+1}\right]+nRY_n\right]\right\}+ \qquad \begin{matrix}(5:52)\\(\text{cont})\end{matrix}$$

$$+i\gamma\left\{\left[-\gamma\bar{b}\left[R\alpha^{(i)}IY_{n+1}+I\alpha^{(i)}RY_{n+1}\right]+IY_n\right]\right\},$$

$$C_{81}=-n\gamma RJ_n-in\gamma IJ_n,$$

$$C_{82}=-n\gamma RY_n-in\gamma IY_n,$$

$$C_{91}=\frac{\gamma^2\bar{b}}{d_5^{(i)}+d_4^{(i)}}\left[RF_2^{(i)}RJ_n-IF_2^{(i)}IJ_n\right]+i\frac{\gamma^2\bar{b}}{d_5^{(i)}+d_4^{(i)}}\left[RF_2^{(i)}IJ_n+IF_2^{(i)}RJ_n\right],$$

$$C_{92}=\frac{\gamma^2\bar{b}}{d_5^{(i)}+d_4^{(i)}}\left[RF_2^{(i)}RY_n-IF_2^{(i)}IY_n\right]+i\frac{\gamma^2\bar{b}}{d_5^{(i)}+d_4^{(i)}}\left[RF_2^{(i)}IY_n+IF_2^{(i)}RY_n\right].$$

The argument of all Bessel functions in (5:52) is $\gamma\bar{a}$ in rows 1, 2, 3, and $\gamma\bar{b}$ in rows 4 through 9. The complex elements for the remaining columns of the frequency equation can be formed according to the instructions contained in (5:37) when applied to (5:52) bearing in mind that

$$C_{ij}=C_{ij}+iC_{(i,j+2)},\quad (i=1,2,\ldots,9;\ j=1,2,7,8,13,14). \qquad (5:53)$$

Equations (5:45), (5:47), and (5:52) taken in conjunction with (5:37) provide the complete specification of the elements of the frequency equation (5:34). Additionally, in order to treat the special case $(d_5^{(i)}+d_4^{(i)})=0$, it is only necessary to modify the i-th material columns of the

frequency equation by the limit given in (3:53). Complying with this limit, all terms of the elements of (5:34) containing $(d_5^{(1)} + d_4^{(0)})$ in either numerator or denominator vanish.

CHAPTER VI

The frequency equation obtained in Chapter V may be altered quite simply in several specific ways to obtain the frequency equations for other useful cases. In this chapter the general frequency equation for the three layered, traction free cylinder will be simplified for the cases of axisymmetry ($n = 0$), infinite axial wave length ($\zeta = 0$), and the combination of both $n = 0$ and $\zeta = 0$. The various steps necessary to obtain the frequency equations for cylinders with clamped (rigidly held) surfaces, and cylinders with combined conditions of free end clamped surfaces will be derived. The number of layers in the cylinder will be generalized, and the conditions to be met in case the inner radius passes to zero will be indicated. Furthermore, the condition of thin outer layers will be considered.

1. Frequency Equation for Three Layered, Traction Free, Isotropic Cylinder

Although equation (5:34) passes over to the isotropic case automatically when $d_3^{(i)} \to d_1^{(i)}$, $d_5^{(i)} \to d_2^{(i)}$, and $d_4^{(i)} \to d_6^{(i)}$, the elements of the determinant will be listed below due to the fundamental importance of the isotropic elastic theory. This listing is accomplished prior to tasks set by the opening paragraph of this chapter so that both the anisotropic and

isotropic equations will be available for comparison in the ensuing simplifications and modifications.

Under the transition to isotropy the radial wave numbers take on the simplified formulations

$$\begin{aligned} p^{(i)2} &= \gamma^2\left|\frac{Z^2}{b^{(i)2}}-1\right| \epsilon_1^{(i)} = \epsilon_1^{(i)}\gamma^2\alpha^{(i)2}, \\ q^{(i)2} &= \gamma^2\left|\frac{Z^2}{b^{(i)2}}-1\right| \epsilon_2^{(i)} = \epsilon_2^{(i)}\gamma^2\beta^{(i)2}, \\ \zeta^{(i)2} &= p^{(i)2} = \gamma^2\left|\frac{Z^2}{b^{(i)2}}-1\right| \end{aligned} \tag{6:1}$$

The factors in (5:37) containing sums of elastic constants and radial wave numbers reduce to

$$\begin{aligned} & d_5^{(i)} + d_4^{(i)} = \frac{1}{2}(d_1^{(i)} + d_2^{(i)}), \\ & d_1^{(i)}\epsilon_1^{(i)}\alpha^{(i)2} + d_3^{(i)}\frac{Z^2}{b^{(i)2}} - d_5^{(i)} = (d_1^{(i)} + d_2^{(i)})\epsilon_1^{(i)}\alpha^{(i)2}, \\ & d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_3^{(i)}\frac{Z^2}{b^{(i)2}} - d_5^{(i)} = \frac{1}{2}(d_1^{(i)} + d_2^{(i)})(\epsilon_1^{(i)}\alpha^{(i)2} - 1), \\ & d_1^{(i)}\epsilon_1^{(i)}\alpha^{(i)2} - d_3^{(i)} - d_4^{(i)}\frac{Z^2}{b^{(i)2}} = \frac{1}{2}(d_1^{(i)} + d_2^{(i)})(\epsilon_1^{(i)}\alpha^{(i)2} - 1), \\ & d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_5^{(i)} - d_4^{(i)}\frac{Z^2}{b^{(i)2}} = -(d_1^{(i)} + d_2^{(i)}), \\ & d_1^{(i)}\epsilon_1^{(i)}\alpha^{(i)2} + d_4^{(i)} - d_4^{(i)}\frac{Z^2}{b^{(i)2}} = \frac{1}{2}(d_1^{(i)} + d_2^{(i)})\epsilon_1^{(i)}\alpha^{(i)2}, \\ & d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_4^{(i)} - d_4^{(i)}\frac{Z^2}{b^{(i)2}} = -\frac{1}{2}(d_1^{(i)} + d_2^{(i)}). \end{aligned} \tag{6:2}$$

Using equations (6:1) and (6:2) in (5:37) results in the following formulation for the elements of the frequency equation:

$$\begin{aligned} C_{ii} &= 2d_6^{(i)}\gamma\left[\epsilon_1^{(i)}\gamma\alpha^{(i)}\bar{\alpha}Z_{n+1}(\gamma\alpha^{(i)}\bar{a}) + \left\{n(n-1) - \epsilon_1^{(i)}\gamma^2\alpha^{(i)2}\bar{a}^2\right\}Z_n(\gamma\alpha^{(i)}\bar{a})\right], \\ C_{ij} &= 2d_6^{(i)}\gamma\left[\gamma\alpha^{(i)}\bar{\alpha}W_{n+1}(\gamma\alpha^{(i)}\bar{a}) + \left\{n(n-1) - \epsilon_1^{(i)}\gamma^2\alpha^{(i)2}\bar{a}^2\right\}W_n(\gamma\alpha^{(i)}\bar{a})\right], \end{aligned} \tag{6:3}$$

$$C_{13} = d_c^{(i)}\left[z\,\epsilon_2^{(i)}\gamma_{\beta^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) + \left\{z\,n(n-1) - \gamma^2\bar{\alpha}^2\,(\epsilon_1^{(i)}\alpha^{(i)2} - 1)\right\}Z_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right],$$

$$C_{14} = d_c^{(i)}\left[z\,\gamma_{\beta^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) + \left\{z\,n(n-1) - \gamma^2\bar{\alpha}^2\,(\epsilon_1^{(i)}\alpha^{(i)2} - 1)\right\}W_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right],$$

$$C_{15} = z\,n\,d_6^{(i)}\left[-\epsilon_1^{(i)}\gamma_{\alpha^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + (n-1)Z_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{16} = z\,n\,d_6^{(i)}\left[\gamma_{\alpha^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + (n-1)W_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{21} = \gamma\,n\,d_6^{(i)}\left[\epsilon_1^{(i)}\gamma_{\alpha^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) - (n-1)Z_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{22} = \gamma\,n\,d_6^{(i)}\left[\gamma_{\alpha^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) - (n-1)W_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{23} = n\,d_6^{(i)}\left[\epsilon_2^{(i)}\gamma_{\beta^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) - (n-1)Z_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right],$$

$$C_{24} = n\,d_6^{(i)}\left[\gamma_{\beta^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) - (n-1)\,W_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right], \qquad (6{:}3)\ (\text{cont})$$

$$C_{25} = -d_6^{(i)}\left[\epsilon_1^{(i)}\gamma_{\alpha^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + \tfrac{1}{2}\left\{z\,n(n-1) - \epsilon_1^{(i)}\gamma^2\alpha^{(i)2}\bar{\alpha}^2\right\}Z_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{26} = -d_6^{(i)}\left[\gamma_{\alpha^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + \tfrac{1}{2}\left\{z\,n(n-1) - \epsilon_1^{(i)}\gamma^2\alpha^{(i)2}\bar{\alpha}^2\right\}W_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{31} = d_6^{(i)}\bar{\alpha}\,\gamma^2(\epsilon_1^{(i)}\alpha^{(i)2} - 1)\left[-\epsilon_1^{(i)}\gamma_{\alpha^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + n\,Z_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{32} = d_6^{(i)}\bar{\alpha}\,\gamma^2(\epsilon_1^{(i)}\alpha^{(i)2} - 1)\left[-\gamma_{\alpha^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\alpha^{(i)}}\bar{\alpha}) + n\,W_n(\gamma_{\alpha^{(i)}}\bar{\alpha})\right],$$

$$C_{33} = -2\,d_6^{(i)}\bar{\alpha}\,\gamma\left[-\epsilon_2^{(i)}\gamma_{\beta^{(i)}}\bar{\alpha}\,Z_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) + n\,Z_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right],$$

$$C_{34} = -2\,d_6^{(i)}\bar{\alpha}\,\gamma\left[-\gamma_{\beta^{(i)}}\bar{\alpha}\,W_{n+1}(\gamma_{\beta^{(i)}}\bar{\alpha}) + n\,W_n(\gamma_{\beta^{(i)}}\bar{\alpha})\right],$$

$$C_{35} = -\bar{\alpha}\,\gamma\,n\,d_6^{(i)}\,Z_n(\gamma_{\alpha^{(i)}}\bar{\alpha}),$$

$$C_{36} = -\bar{\alpha}\,\gamma\,n\,d_6^{(i)}\,W_n(\gamma_{\alpha^{(i)}}\bar{\alpha}),$$

$$C_{71} = \gamma\left[-\epsilon_1^{(i)}\gamma_{\alpha^{(i)}}b\,Z_{n+1}(\gamma_{\alpha^{(i)}}t_6) + n\,Z_n(\gamma_{\alpha^{(i)}}t_6)\right],$$

$$C_{72} = \gamma\left[-\gamma_{\alpha^{(i)}}t_6\,W_{n+1}(\gamma_{\alpha^{(i)}}t_6) + n\,W_n(\gamma_{\alpha^{(i)}}t_6)\right],$$

$$C_{73} = \left[-\epsilon_1^{(i)} \gamma \beta^{(i)} b \, Z_{n+1}(\gamma \beta^{(i)} b) + n Z_n(\gamma \beta^{(i)} b)\right],$$

$$C_{74} = \left[-\gamma \beta^{(i)} b \, W_{n+1}(\gamma \beta^{(i)} b) + n W_n(\gamma \beta^{(i)} b)\right],$$

$$C_{75} = n Z_n(\gamma \alpha^{(i)} b),$$

$$C_{76} = n W_n(\gamma \alpha^{(i)} b),$$

$$C_{81} = -n \gamma Z_n(\gamma \alpha^{(i)} b),$$

$$C_{82} = -n \gamma W_n(\gamma \alpha^{(i)} b),$$

$$C_{83} = -n Z_n(\gamma \beta^{(i)} b),$$

$$C_{84} = -n W_n(\gamma \beta^{(i)} b), \qquad (6:3) \text{ (cont)}$$

$$C_{85} = \epsilon_1^{(i)} \gamma \alpha^{(i)} b \, Z_{n+1}(\gamma \alpha^{(i)} b) - n Z_n(\gamma \alpha^{(i)} b),$$

$$C_{86} = \gamma \alpha^{(i)} b \, W_{n+1}(\gamma \alpha^{(i)} b) - n W_n(\gamma \alpha^{(i)} b),$$

$$C_{91} = \epsilon_1^{(i)} \gamma^2 \alpha^{(i)2} b \, Z_n(\gamma \alpha^{(i)} b),$$

$$C_{92} = \epsilon_1^{(i)} \gamma^2 \alpha^{(i)2} b \, W_n(\gamma \alpha^{(i)} b),$$

$$C_{93} = -\gamma b \, Z_n(\gamma \beta^{(i)} b),$$

$$C_{94} = -\gamma b \, W_n(\gamma \beta^{(i)} b),$$

$$C_{95} = C_{96} = 0.$$

The remainder of the elements of the frequency equation are formed from equations (6:3) and the instructions of equation (5:37).

The turning point line expressions of Chapter V are

directly applicable to the isotropic equations when the proper portions of (6:2) are employed. The scope of their applicability, however, is limited to that for less anisotropic, class I materials. This means that $q^{(i)}$ vanishes only when $Z = \mathfrak{b}^{(i)}$, and $s^{(i)}$ and $p^{(i)}$, being now the same, always vanish when $Z = b^{(i)}$. The minor changes to be effected in (5:45) to make them applicable to isotropy do not seem to warrant further consideration at this point.

2. Frequency Equation for Waves of Infinite Length

When the axial wavelength of the traveling waves approaches infinity, the axial wave number approaches zero, and the frequency equation becomes a function of Ω alone. For this case the values of Ω for which (5:34) does indeed vanish are called cut-off frequencies.

Using the appropriate forms of the radial wave numbers when $\zeta = 0$, that is, equations (4:11) and (4:48), and using the limit expressions for $p_c^{(i)2}$ and $q_c^{(i)2}$ where necessary, the following terms from (5:37) are noted below for convenience.

$$
\begin{aligned}
&d_1^{(i)}\rho^{(i)2} + d_5^{(i)}\frac{\Omega^2}{b^{(i)2}} - d_5^{(i)}\zeta^2 \to \left(d_1^{(i)} + d_5^{(i)}\right)\frac{\Omega^2}{b^{(i)2}}, \\
&d_4^{(i)}\sigma^{(i)2} + d_5^{(i)}\frac{\Omega^2}{b^{(i)2}} - d_5^{(i)}\zeta^2 \to \left(d_4^{(i)} + d_5^{(i)}\right)\frac{\Omega^2}{b^{(i)2}}, \\
&d_1^{(i)}\rho^{(i)2} - d_4^{(i)}\frac{\Omega^2}{b^{(i)2}} - d_1^{(i)}\zeta^2 \to \left(d_1^{(i)} - d_4^{(i)}\right)\frac{\Omega^2}{b^{(i)2}}, \\
&\frac{1}{\zeta}\left(d_1^{(i)}\sigma^{(i)2} - d_4^{(i)}\frac{\Omega^2}{b^{(i)2}} - d_5^{(i)}\zeta^2\right) \to 0, \\
&d_1^{(i)}\rho^{(i)2} - d_4^{(i)}\frac{\Omega^2}{b^{(i)2}} + d_4^{(i)}\zeta^2 \to \left(d_1^{(i)} - d_4^{(i)}\right)\frac{\Omega^2}{b^{(i)2}}, \\
&\frac{1}{\zeta}\left(d_1^{(i)}\sigma^{(i)2} - d_4^{(i)}\frac{\Omega^2}{b^{(i)2}} + d_4^{(i)}\zeta^2\right) \to 0
\end{aligned}
\tag{6:4}
$$

When $\gamma = 0$ the radial wave numbers take the form:

$$\begin{aligned} p_c^{(i)} &= \Omega^2 / \beta_{12}^{(i)2}, \\ q_c^{(i)} &= d_4^{(i)} p_c^{(i)} / \beta_1^{(i)}, \\ s_c^{(i)} &= d_4^{(i)} p_c^{(i)} / d_c^{(i)} \end{aligned} \tag{6:5}$$

Finally, since $p_c^{(i)}$, $q_c^{(i)}$, and $s_c^{(i)}$ are in sector 7, only regular Bessel functions will appear in the frequency equation. Using (6:4) and (6:5) in (5:37) results in

$$c_{11} = c_{12} = 0,$$

$$c_{13} = 2 d_c^{(i)} q_c^{(i)} \bar{a} J_{n+1}(q_c^{(i)} \bar{a}) + \left[2 d_c^{(i)} n(n-1) - d_4^{(i)2} \bar{a}^2 \frac{\Omega^2}{\beta_{12}^2} \right] J_n(q_c^{(i)} \bar{a}),$$

$$c_{14} = 2 d_c^{(i)} q_c^{(i)} \bar{a} Y_{n+1}(q_c^{(i)} \bar{a}) + \left[2 d_c^{(i)} n(n-1) - d_4^{(i)2} \bar{a}^2 \frac{\Omega^2}{\beta_{12}^2} \right] Y_n(q_c^{(i)} \bar{a}),$$

$$c_{15} = 2 n d_c^{(i)} \left[-s_c^{(i)} \bar{a} J_{n+1}(s_c^{(i)} \bar{a}) + (n-1) J_n(s_c^{(i)} \bar{a}) \right],$$

$$c_{16} = 2 n d_c^{(i)} \left[-s_c^{(i)} \bar{a} Y_{n+1}(s_c^{(i)} \bar{a}) + (n-1) Y_n(s_c^{(i)} \bar{a}) \right],$$

$$c_{21} = c_{22} = 0,$$

$$c_{23} = n d_c^{(i)} \left[q_c^{(i)} \bar{a} J_{n+1}(q_c^{(i)} \bar{a}) - (n-1) J_n(q_c^{(i)} \bar{a}) \right],$$

$$c_{24} = n d_c^{(i)} \left[q_c^{(i)} \bar{a} Y_{n+1}(q_c^{(i)} \bar{a}) - (n-1) Y_n(q_c^{(i)} \bar{a}) \right], \tag{6:6}$$

$$c_{25} = -d_c^{(i)} s_c^{(i)} \bar{a} J_{n+1}(s_c^{(i)} \bar{a}) - d_c^{(i)} \left[n(n-1) - s_c^{(i)2} \bar{a}^2 / 2 \right] J_n(s_c^{(i)} \bar{a}),$$

$$c_{26} = -d_c^{(i)} s_c^{(i)} \bar{a} Y_{n+1}(s_c^{(i)} \bar{a}) - d_c^{(i)} \left[n(n-1) - s_c^{(i)2} \bar{a}^2 / 2 \right] Y_n(s_c^{(i)} \bar{a}),$$

$$c_{31} = d_4^{(i)} \left[-p_c^{(i)} \bar{a} J_{n+1}(p_c^{(i)} \bar{a}) + n J_n(p_c^{(i)} \bar{a}) \right],$$

$$c_{32} = d_4^{(i)} \left[-p_c^{(i)} \bar{a} Y_{n+1}(p_c^{(i)} \bar{a}) + n Y_n(p_c^{(i)} \bar{a}) \right],$$

$$c_{33} = c_{34} = c_{35} = c_{36},$$

$$C_{71} = C_{72} = 0,$$

$$C_{73} = -q_c^{(1)} b\, J_{n+1}(q_c^{(1)} b) + n J_n(q_c^{(1)} b),$$

$$C_{74} = -q_c^{(1)} b\, Y_{n+1}(q_c^{(1)} b) + n Y_n(q_c^{(1)} b),$$

$$C_{75} = n J_n(s_c^{(1)} b),$$

$$C_{76} = n Y_n(s_c^{(1)} b),$$

$$C_{81} = C_{82} = 0,$$

$$C_{83} = -n J_n(q_c^{(1)} b), \qquad (6:6)\ \text{(cont)}$$

$$C_{84} = -n Y_n(q_c^{(1)} b),$$

$$C_{85} = s_c^{(1)} b\, J_{n+1}(s_c^{(1)} b) - n J_n(s_c^{(1)} b),$$

$$C_{86} = s_c^{(1)} b\, Y_{n+1}(s_c^{(1)} b) - n Y_n(s_c^{(1)} b),$$

$$C_{91} = J_n(p_c^{(1)} b),$$

$$C_{92} = Y_n(p_c^{(1)} b),$$

$$C_{93} = C_{94} = C_{95} = C_{96} = 0.$$

The remaining elements of the determinant are formed from the above terms according to the instructions in equation (5:37).

Using the elements listed in (6:6), the frequency equation decomposes into the following product of determinants in the limit as $\gamma \to 0$:

$$\lim_{\gamma \to 0} \det |C_{ij}| = D_1 D_2 = 0, \qquad (6:7)$$

where

$$D_1 = \begin{vmatrix} C_{31} & C_{32} & 0 & 0 & 0 & 0 \\ C_{41} & C_{42} & C_{47} & C_{48} & 0 & 0 \\ C_{91} & C_{92} & C_{97} & C_{98} & 0 & 0 \\ 0 & 0 & C_{10,7} & C_{10,8} & C_{10,13} & C_{10,14} \\ 0 & 0 & C_{15,7} & C_{15,8} & C_{15,13} & C_{15,14} \\ 0 & 0 & 0 & 0 & C_{16,13} & C_{16,14} \end{vmatrix}, \qquad (a)$$

and (6:8)

$$D_2 = \begin{vmatrix} C_{13} & C_{14} & C_{15} & C_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{23} & C_{24} & C_{25} & C_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{53} & C_{54} & C_{55} & C_{56} & C_{59} & C_{5,10} & C_{5,11} & C_{5,12} & 0 & 0 & 0 & 0 \\ C_{63} & C_{64} & C_{65} & C_{66} & C_{69} & C_{6,10} & C_{6,11} & C_{6,12} & 0 & 0 & 0 & 0 \\ C_{73} & C_{74} & C_{75} & C_{76} & C_{79} & C_{7,10} & C_{7,11} & C_{7,12} & 0 & 0 & 0 & 0 \\ C_{83} & C_{84} & C_{85} & C_{86} & C_{89} & C_{8,10} & C_{8,11} & C_{8,12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{11,9} & C_{11,10} & C_{11,11} & C_{11,12} & C_{11,15} & C_{11,16} & C_{11,17} & C_{11,18} \\ 0 & 0 & 0 & 0 & C_{12,9} & C_{12,10} & C_{12,11} & C_{12,12} & C_{12,15} & C_{12,16} & C_{12,17} & C_{12,18} \\ 0 & 0 & 0 & 0 & C_{13,9} & C_{13,10} & C_{13,11} & C_{13,12} & C_{13,15} & C_{13,16} & C_{13,17} & C_{13,18} \\ 0 & 0 & 0 & 0 & C_{14,9} & C_{14,10} & C_{14,11} & C_{14,12} & C_{14,15} & C_{14,16} & C_{14,17} & C_{14,18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{17,15} & C_{17,16} & C_{17,17} & C_{17,18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{18,15} & C_{18,16} & C_{18,17} & C_{18,18} \end{vmatrix}, \qquad (b)$$

The limit process eliminates from the displacement field all dependence on the axial coordinate z and also uncouples the effects of $\varphi_1^{(i)}$ from $f^{(i)}$ and $\varphi_2^{(i)}$ (i = 1,2,3). This can be shown as follows. If $D_1 = 0$, only the integration constants $A_1^{(i)}$, $B_1^{(i)}$ (i = 1,2,3) are nonzero and equations (5:23) yield

$$\varphi_1^{(i)} = A_1^{(i)} J_0(\xi_1^{(i)} \bar{r}) + B_1^{(i)} Y_0(\xi_1^{(i)} \bar{r}), \qquad (6:9)$$

$$\phi^{(i)} = \psi_1^{(i)} = 0 \qquad (6:9)$$

(cont)

Using (6:9) in (5:24) gives

$$u_r^{(i)} = u_\theta^{(i)} = 0,$$
$$u_z^{(i)} = -\frac{(\omega^{(i)} - d_4^{(i)})\beta^{(i)}}{(\omega_3^{(i)} + d_4^{(i)})\eta} \hat{r}_c^{(i)2} \left(A_i^{(i)} J_n(\hat{r}_c^{(i)}\bar{r}) + B_i^{(i)} Y_n(\hat{r}_c^{(i)}\bar{r}) \right) \cos n\theta \sin \Omega\bar{t} \qquad (6:10)$$

$D_1 = 0$ is, consequently, the frequency equation for longitudinal thickness-shear vibrations. Only the elastic constants $d_4^{(i)}$ (i = 1,2,3) are involved in D_1. This is to be expected since a review of the stress state reveals that only τ_4 and τ_5 are nonzero. In turn, these two stresses are related to strains only through the $d_4^{(i)}$ (i = 1,2,3).

On the other hand, if $D_2 = 0$, then $A_j^{(i)}$, $B_j^{(i)}$ (i = 1,2,3; j = 2,3) are nonzero, and (5:23) yields

$$\phi_2^{(i)} = A_2^{(i)} J_n(\hat{r}_\mu^{(i)}\bar{r}) + B_2^{(i)} Y_n(\hat{r}_\mu^{(i)}\bar{r}) \quad , \quad \psi_1^{(i)} = 0,$$
$$\psi^{(i)} = A_3^{(i)} J_n(\hat{r}_c^{(i)}\bar{r}) + B_3^{(i)} Y_n(\hat{r}_c^{(i)}\bar{r}) \,. \qquad (6:11)$$

Equations (6:11), in turn, produce the following displacement field:

$$u_r^{(i)} = \frac{\beta^{(i)}}{\eta} \left\{ (\omega_3^{(i)} + d_4^{(i)}) \left[A_2^{(i)} \frac{d J_n(\hat{r}_\mu^{(i)}\bar{r})}{d\bar{r}} + B_2^{(i)} \frac{d Y_n(\hat{r}_\mu^{(i)}\bar{r})}{d\bar{r}} \right] + \right.$$
$$\left. + \frac{\eta}{\bar{r}} \left[A_3^{(i)} J_n(\hat{r}_c^{(i)}\bar{r}) + B_3^{(i)} Y_n(\hat{r}_c^{(i)}\bar{r}) \right] \right\} \cos n\theta \cos \Omega\bar{t} \,,$$
$$u_\theta^{(i)} = -\frac{\beta^{(i)}}{\eta} \left\{ (\omega_3^{(i)} + d_4^{(i)}) \left[A_2^{(i)} J_n(\hat{r}_\mu^{(i)}\bar{r}) + B_2^{(i)} Y_n(\hat{r}_\mu^{(i)}\bar{r}) \right] + \right. \qquad (6:12)$$
$$\left. + A_3^{(i)} \frac{d J_n(\hat{r}_c^{(i)}\bar{r})}{d\bar{r}} + B_3^{(i)} \frac{d Y_n(\hat{r}_c^{(i)}\bar{r})}{d\bar{r}} \right] \sin n\theta \cos \Omega\bar{t},$$
$$u_z^{(i)} = 0.$$

$D_2 = 0$ is the frequency equation for plane strain vibrations. Examination of the elements of D_2 shows that the motion is a combination of equivoluminal and dilatational effects.

Moreover, all of the elastic constants except $d_3^{(i)}$ and $d_5^{(i)}$ are involved in the determinant. This is to be expected since the stress-strain relations in Chapter II show that when $s_3^{(i)}=0$ the elastic constants $c_{13}^{(i)}(=c_{44}^{(i)}d_5^{(i)})$ and $c_{33}^{(i)}(=c_{44}^{(i)}d_3^{(i)})$ play no rôle in the determination of stresses.

3. Frequency Equation for Axisymmetric Traveling Waves

To eliminate the angular coordinate from the displacement equations the integer n may be set equal to zero. When this is accomplished certain simplifications occrus to the formulation of the frequency equation. Using equation (5:37), the elements of the frequency equation determinant can immediately be written as

$$c_{11} = 2d_6^{(i)}\epsilon_1^{(i)}\gamma^2\xi^{(i)}\bar{\alpha}\,Z_1(\gamma\xi^{(i)}\bar{\alpha}) - \frac{d_6^{(i)}\gamma^3\bar{\alpha}^2}{d_5^{(i)}+d_6^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]Z_0(\gamma\xi^{(i)}\bar{\alpha}),$$

$$c_{12} = 2d_6^{(i)}\gamma^2\xi^{(i)}\bar{\alpha}\,W_1(\gamma\xi^{(i)}\bar{\alpha}) - \frac{d_6^{(i)}\gamma^3\bar{\alpha}^2}{d_5^{(i)}+d_6^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]W_0(\gamma\xi^{(i)}\bar{\alpha}),$$

$$c_{13} = 2d_6^{(i)}(d_5^{(i)}+d_4^{(i)})\epsilon_2^{(i)}\gamma\beta^{(i)}\bar{\alpha}\,Z_1(\gamma\beta^{(i)}\bar{\alpha}) - d_4^{(i)}\gamma^2\bar{\alpha}^2\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]Z_0(\gamma\beta^{(i)}\bar{\alpha}),$$

$$c_{14} = 2d_6^{(i)}(d_5^{(i)}+d_4^{(i)})\gamma\beta^{(i)}\bar{\alpha}\,W(\gamma\beta^{(i)}\bar{\alpha}) - d_4^{(i)}\gamma^2\bar{\alpha}^2\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_5^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]W_0(\gamma\beta^{(i)}\bar{\alpha}),$$

$$c_{15} = c_{16} = 0,$$

(6:13)

$$c_{21} = c_{22} = c_{23} = c_{24} = 0,$$

$$c_{25} = -\frac{1}{2}d_6^{(i)}\gamma^2\xi^{(i)2}\bar{\alpha}^2\,Z_2(\gamma\xi^{(i)}\bar{\alpha}),$$

$$c_{26} = -\frac{1}{2}d_6^{(i)}\gamma^2\xi^{(i)2}\bar{\alpha}^2\,W_2(\gamma\xi^{(i)}\bar{\alpha}),$$

$$c_{31} = \frac{-d_4^{(i)}\gamma^2\bar{\alpha}}{d_5^{(i)}+d_6^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\epsilon_1^{(i)}\gamma\xi^{(i)}\bar{\alpha}\,Z_1(\gamma\xi^{(i)}\bar{\alpha}),$$

$$c_{32} = \frac{-d_4^{(i)}\gamma^2\bar{\alpha}}{d_5^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\gamma\xi^{(i)}\bar{\alpha}\,W_1(\gamma\xi^{(i)}\bar{\alpha}),$$

$$C_{33} = -d_4^{(1)}\gamma_1 a\left[d_1^{(1)}\varepsilon_2^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} - d_5^{(1)}\right]\varepsilon_2^{(1)}\gamma_1\beta^{(1)}\bar{a}\,Z_1(\gamma_1\beta^{(1)}\bar{a}),$$

$$C_{34} = -d_4^{(1)}\gamma_1 a\left[d_1^{(1)}\varepsilon_2^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} - d_5^{(1)}\right]\gamma_1\beta^{(1)}\bar{a}\,W_1(\gamma_1\beta^{(1)}\bar{a}),$$

$$C_{35} = C_{36} = 0,$$

$$C_{71} = -\varepsilon_1^{(1)}\gamma_1^2\,\alpha^{(1)}\,b\,Z_1(\gamma_1\alpha^{(1)}b),$$

$$C_{72} = -\gamma_1^2\,\alpha^{(1)}\,b\,W_1(\gamma_1\alpha^{(1)}b),$$

$$C_{73} = -(d_5^{(1)} + d_4^{(1)})\varepsilon_2^{(1)}\gamma_1\beta^{(1)}\,b\,Z_1(\gamma_1\beta^{(1)}b),$$

$$C_{74} = -(d_5^{(1)} + d_4^{(1)})\gamma_1\beta^{(1)}\,b\,W_1(\gamma_1\beta^{(1)}b),$$

$$C_{75} = C_{76} = 0$$

$$C_{81} = C_{82} = C_{84} = 0,$$ (6:13) (cont)

$$C_{85} = \varepsilon_2^{(1)}\gamma_2\,b\,Z_1(\gamma_2\beta^{(1)}b),$$

$$C_{86} = \gamma_2\,b\,W_1(\gamma_2\beta^{(1)}b),$$

$$C_{91} = \frac{\Sigma^2 b}{d_5^{(1)} + d_4^{(1)}}\left[d_1^{(1)}\varepsilon_1^{(1)}\alpha^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)}\right]Z_0(\gamma_1\alpha^{(1)}b),$$

$$C_{92} = \frac{\Sigma^2 b}{d_5^{(1)} + d_4^{(1)}}\left[d_1^{(1)}\varepsilon_1^{(1)}\alpha^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)}\right]W_0(\gamma_1\alpha^{(1)}b),$$

$$C_{93} = \gamma_2 b\left[d_1^{(1)}\varepsilon_2^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)}\right]Z_0(\gamma_2\beta^{(1)}b),$$

$$C_{94} = \gamma_2 b\left[d_1^{(1)}\varepsilon_2^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)}\right]W_0(\gamma_2\beta^{(1)}b),$$

$$C_{95} = C_{96} = 0.$$

The remaining elements of the determinant are formed from the above terms according to the instructions in equation (5:37).

Using equations (6:13) in the frequency equation produces the following product of determinants

$$\lim_{n \to 0} \det |C_{ij}| = D_3 \cdot D_4 = 0, \tag{6:14}$$

where

$$D_3 = \begin{vmatrix} C_{25} & C_{26} & 0 & 0 & 0 & 0 \\ C_{55} & C_{56} & C_{5,11} & C_{5,12} & 0 & 0 \\ C_{85} & C_{86} & C_{8,11} & C_{8,12} & 0 & 0 \\ 0 & 0 & C_{11,11} & C_{11,12} & C_{11,17} & C_{11,18} \\ 0 & 0 & C_{14,11} & C_{14,12} & C_{14,17} & C_{14,18} \\ 0 & 0 & 0 & 0 & C_{17,17} & C_{17,18} \end{vmatrix}, \tag{a}$$

and (6:15)

$$D_4 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{47} & C_{48} & C_{49} & C_{4,10} & 0 & 0 & 0 & 0 \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{67} & C_{68} & C_{69} & C_{6,10} & 0 & 0 & 0 & 0 \\ C_{71} & C_{72} & C_{73} & C_{74} & C_{77} & C_{78} & C_{79} & C_{7,10} & 0 & 0 & 0 & 0 \\ C_{91} & C_{92} & C_{93} & C_{94} & C_{97} & C_{98} & C_{99} & C_{9,10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} & C_{10,13} & C_{10,14} & C_{10,15} & C_{10,16} \\ 0 & 0 & 0 & 0 & C_{12,7} & C_{12,8} & C_{12,9} & C_{12,10} & C_{12,13} & C_{12,14} & C_{12,15} & C_{12,16} \\ 0 & 0 & 0 & 0 & C_{13,7} & C_{13,8} & C_{13,9} & C_{13,10} & C_{13,13} & C_{13,14} & C_{13,15} & C_{13,16} \\ 0 & 0 & 0 & 0 & C_{15,7} & C_{15,8} & C_{15,9} & C_{15,10} & C_{15,13} & C_{15,14} & C_{15,15} & C_{15,16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{16,13} & C_{16,14} & C_{16,15} & C_{16,16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{18,13} & C_{18,14} & C_{18,15} & C_{18,16} \end{vmatrix}. \tag{b}$$

If $D_3 = 0$, the constants $A_3^{(i)}$ and $B_3^{(i)}$ $(i = 1,2,3)$ are not generally zero. Thus, (5:23) yields

$$\varphi_1^{(i)} = \varphi_{12}^{(i)} = 0,$$

$$\psi^{(i)} = A_3^{(i)} Z_0(\tau \xi^{(i)} \bar{r}) + B_3^{(i)} W_0(\tau \xi^{(i)} \bar{r}) \tag{6:16}$$

With regard to equation (3:57) the displacement field resulting from the requirement that $D_3 = 0$ is given as

$$u_r^{(i)} = u_z^{(i)} = 0,$$
$$u_\theta^{(i)} = \frac{\beta^{(i)}}{\pi}\left[A_3^{(i)}\varepsilon_3^{(i)} Z_1(\zeta_i^{(i)}\bar{r}) + B_3^{(i)} W_1(\zeta_i^{(i)}\bar{r})\right]\zeta_i^{(i)} \sin(\zeta\bar{z} - \Omega\bar{t}). \tag{6:17}$$

Thus, the frequency equation $D_3 = 0$ corresponds to the family of axisymmetric modes of traveling waves generated solely by the $f^{(i)}(\bar{r})$ $(i = 1,2,3)$ potentials. The displacement field is torsional in character and has nodal cylinders for $\zeta_i^{(i)}\bar{r}$ sufficiently large. Consideration of the elements in D_3 shows a dependence on both $d_4^{(i)}$ and $d_6^{(i)}$ which suggests the mechanisms of tangential and radial shear employed to resist torsion. When the limit is taken to isotropic conditions, $d_4^{(i)} \to d_6^{(i)}$, and the rôle of $\mu^{(i)}(=d_6^{(i)})$ is clearly revealed to derive from the two separate anisotropic mechanisms.

$D_4 = 0$ is the frequency equation for nontorsional axisymmetric wave motion. Only the integration constants $A_3^{(i)}$ and $B_3^{(i)}$ $(i = 1,2,3)$ are zero, consequently, equation (5:23) yields

$$\varphi^{(i)}(\bar{r}) = \frac{\sum}{d_5^{(i)} + d_4^{(i)}} \frac{\pi}{\beta^{(i)} c_{44}^{(i)}} \left(\tilde{A}_1^{(i)} Z_0(\zeta_i^{(i)}\bar{r}) + B_1^{(i)} W_0(\zeta_i^{(i)}\bar{r})\right) +$$
$$+ \tilde{A}_2^{(i)} Z_0(\zeta_i^{(i)}\bar{r}) + \tilde{B}_2^{(i)} W_0(\zeta_i^{(i)}\bar{r}), \tag{6:18}$$
$$\psi^{(i)}(\bar{r}) = 0$$

Using equation (5:24), the displacement field is given

$$u_r^{(i)} = \frac{\beta^{(i)}}{\pi}\left\{-\zeta^2 \varepsilon^{(i)}\left(\varepsilon_1^{(i)} A_1^{(i)} Z_1(\zeta_i^{(i)}\bar{r}) + B_1^{(i)} W_1(\zeta_i^{(i)}\bar{r})\right) -\right.$$
$$\left. - (d_5^{(i)} + d_4^{(i)})\zeta_i^{(i)}\left(\varepsilon_2^{(i)} A_2^{(i)} Z_1(\zeta_i^{(i)}\bar{r}) + B_2^{(i)} W_1(\zeta_i^{(i)}\bar{r})\right)\right\} \cos(\zeta\bar{z} - \Omega\bar{t}), \tag{6:19}$$
$$u_\theta^{(i)} = 0,$$

$$u_z^{(i)} = \frac{ik^{(i)}}{\alpha}\left\{\frac{\zeta}{\left(\frac{\omega^{(i)}}{\alpha_4^{(i)}}\right)^2+\zeta^2}\left(d_1^{(i)}\epsilon_1^{(i)}\varepsilon_1^{(i)}-d_5^{(i)}\frac{\beta^{(i)}}{k^{(i)}}+d_6^{(i)}\right)\left(A_1^{(i)}Z_0(\gamma_1\varepsilon^{(i)}\bar{r})+B_1^{(i)}W_0(\gamma_1\varepsilon^{(i)}\bar{r})\right)+\right.$$

$$\left.+\left(d_2^{(i)}\epsilon_2^{(i)}\varepsilon^{(i)}-d_4^{(i)}\frac{\beta^{(i)}}{k^{(i)}}+d_4^{(i)}\right)\left(A_2^{(i)}Z_0(\gamma_2\varepsilon^{(i)}\bar{r})+B_2^{(i)}W_0(\gamma_2\varepsilon^{(i)}\bar{r})\right)\right\}\sin(\gamma\bar{z}-\Omega\bar{t}). \quad (6:19)$$

(cont)

The displacement field is seen to be comprised of radial and axial components and is caused separately of the $f^{(i)}(r)$ $(i = 1,2,3)$ potentials. In the limit to isotropy the two sets of potentials $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ $(i = 1,2,3)$ pass over to equivoluminal and dilatational potentials, respectively, and the frequency equation dependence on elastic properties can be related to the three Poisson's ratios $\nu^{(i)}$ $(i = 1,2,3)$. In the anisotropic formulation all of the elastic moduli are present, indicating an involved dependence of the modes of wave motion on the elastic parameters of the layers of the cylinder. This result is to be expected on the basis of the transversely isotropic stress-strain relations.

4. Frequency Equation for Axisymmetric Waves of Infinite Length

If both n and ζ are set equal to zero the frequency equation degenerates to a product of three 6X6 determinants, reflecting the fact that the three sets of potentials $\varphi_1^{(i)}$, $\varphi_2^{(i)}$, and $f^{(i)}$ $(i = 1,2,3)$ generate uncoupled motion in the composite cylinder. The appropriate formulation of the elements of the frequency equation may be derived from (5:37), or either of the last two sections may be used as a starting point. Choosing to proceed with the formulation established in Section 2, the elements C_{1j} are given as

$$C_{11} = C_{12} = 0, \quad (6:20)$$

$$C_{13} = 2 d_c^{(1)} q_c^{(1)} \bar{a} J_1(q_c^{(1)} \bar{a}) - d_4^{(1)} \bar{a}^2 \frac{\Omega^2}{k^{(1)2}} J_0(q_c^{(1)} \bar{a}),$$

$$C_{14} = 2 d_c^{(1)} q_c^{(1)} \bar{a} Y_1(q_c^{(1)} \bar{a}) - d_4^{(1)} \bar{a}^2 \frac{\Omega^2}{k^{(1)2}} Y_0(q_c^{(1)} \bar{a}),$$

$$C_{15} = C_{16} = 0,$$

$$C_{21} = C_{22} = C_{23} = C_{24} = 0,$$

$$C_{25} = -\frac{1}{2} d_6^{(1)} s_c^{(1)2} \bar{a}^2 J_2(s_c^{(1)} \bar{a}),$$

$$C_{26} = -\frac{1}{2} d_6^{(1)} s_c^{(1)2} \bar{a}^2 Y_2(s_c^{(1)} \bar{a}),$$

$$C_{31} = -d_4^{(1)} p_c^{(1)} \bar{a} J_1(p_c^{(1)} \bar{a}),$$

$$C_{32} = -d_4^{(1)} p_c^{(1)} \bar{a} Y_1(p_c^{(1)} \bar{a}),$$

$$C_{33} = C_{34} = C_{35} = C_{36} = 0,$$

$$C_{71} = C_{72} = 0,$$

$$C_{73} = -q_c^{(1)} b J_1(q_c^{(1)} b),$$

$$C_{74} = -q_c^{(1)} b Y_1(q_c^{(1)} b)$$

$$C_{75} = C_{76} = 0,$$

$$C_{81} = C_{82} = C_{83} = C_{84} = 0,$$

$$C_{85} = s_c^{(1)} b J_1(s_c^{(1)} b),$$

$$C_{86} = s_c^{(1)} b Y_1(s_c^{(1)} b),$$

$$C_{91} = J_0(p_c^{(1)} b),$$

$$C_{92} = Y_0(p_c^{(1)} b),$$

(6.20) (cont)

$$C_{53} = C_{54} = C_{55} = C_{56} = 0 \,. \qquad (6:20)\ (\text{cont})$$

The remaining elements of the frequency equation are formed from (6:20) according to the instructions of (5:37).

Employing (6:20) in the frequency equation results in

$$\lim_{\substack{n\to 0\\ \bar{r}\to 0}} \det\left|C_{ij}\right| = \tilde{D}_1 \cdot \tilde{D}_3 \cdot D_5 = 0, \qquad (6:21)$$

where $\tilde{D}_1$ and $\tilde{D}_3$ are formed from the elements of (6:20) according to equations (6:8e) and (6:15e), and D_5 is given as

$$D_5 = \begin{vmatrix} C_{13} & C_{14} & 0 & 0 & 0 & 0 \\ C_{23} & C_{24} & C & C & 0 & 0 \\ C_{73} & C_{74} & C_{75} & C_{7,10} & 0 & 0 \\ 0 & 0 & C_{12,9} & C_{12,10} & C_{12,15} & C_{12,16} \\ 0 & 0 & C_{13,9} & C_{13,10} & C_{13,15} & C_{13,16} \\ 0 & 0 & 0 & 0 & C_{16,15} & C_{16,16} \end{vmatrix} . \qquad (a)\ (6:22)$$

If $\tilde{D}_1 = 0$ only $A_1^{(i)}$, $B_1^{(i)}$ (i = 1,2,3) are nonzero yielding

$$\varphi_1^{(i)} = \tilde{A}_1^{(i)} J_0(\rho_c^{(i)} \bar{r}) + \tilde{B}_1^{(i)} Y_0(\rho_c^{(i)} \bar{r}), \quad f^{(i)} = \varphi_{12}^{(i)} = 0, \qquad (e)$$

and the displacement field

$$u_r^{(i)} = u_\theta^{(i)} = 0, \qquad (b) \qquad (6:23)$$

$$u_z^{(i)} = -\frac{k^{(i)} Y_0^{(i)} - \lambda_0^{(i)}}{\eta_1^{(i)} \left(\rho_c^{(i)2} + \alpha_4^{(i)}\right)} \rho_c^{(i)2} \left(A_1^{(i)} J_0(\rho_c^{(i)} \bar{r}) + B_1^{(i)} Y_0(\rho_c^{(i)} \bar{r})\right) \sin \Omega \bar{t} \,. \qquad (c)$$

$\tilde{D}_1 = 0$ is, thus, the frequency equation for symmetric, longitudinal-shear vibrations, and the displacement field depends only on $\varphi_1^{(i)}(\bar{r})$ (i = 1,2,3).

If $\tilde{D}_3 = 0$ only $A_3^{(i)}$, $B_3^{(i)}$ (i = 1,2,3) need be nonzero, resulting in

$$f^{(i)} = \tilde{A}_3^{(i)} J_0(\xi_c^{(i)} \bar{r}) + \tilde{B}_3^{(i)} Y_0(\xi_c^{(i)} \bar{r}), \quad \varphi_1^{(i)} = \varphi_{12}^{(i)} = 0, \qquad (a)\ (6:24)$$

and

$$u_r^{(i)} = u_z^{(i)} = 0, \qquad \text{(b)}$$

$$u_\theta^{(i)} = -\frac{k^{(i)}}{\eta}\left(A_3^{(i)}\xi_\zeta^{(i)} J_1(\xi_\zeta^{(i)}\bar{r}) + B_3^{(i)}\xi_\zeta^{(i)} Y_1(\xi_\zeta^{(i)}\bar{r})\right)\sin\Omega\bar{t} \qquad \text{(c)} \qquad \text{(6:24 cont)}$$

It is clear that $\lim_{\zeta \to 0} D_3 = \tilde{D}_3$, hence $\tilde{D}_3 = 0$ yields the cut-off frequencies of the axisymmetric torsional waves. Since $\tilde{D}_3 = 0$ produces the plane-strain displacement field (6:24), it follows that the torsional waves approach plane strain-shear vibrations as $\zeta \to 0$.

Finally, with $D_5 = 0$ only $A_2^{(i)}$ and $B_2^{(i)}$ $(i = 1,2,3)$ need be nonzero, yielding

$$\varphi_2^{(i)} = \tilde{A}_2^{(i)} J_0(\xi_\zeta^{(i)}\bar{r}) + \tilde{B}_2^{(i)} Y_0(\xi_\zeta^{(i)}\bar{r}), \quad \varphi_1^{(i)} = f^{(i)} = 0, \qquad \text{(a)}$$

and

$$u_\theta^{(i)} = u_z^{(i)} = 0, \qquad \text{(b)} \qquad \text{(6:25)}$$

$$u_r^{(i)} = -\frac{k^{(i)}}{\eta}\left(\xi_\zeta^{(i)} f^{(i)} + \xi_4^{(i)}\varphi_2^{(i)}\right)\left(A_2^{(i)} J_1(\xi_\zeta^{(i)}\bar{r}) + B_2^{(i)} Y_1(\xi_\zeta^{(i)}\bar{r})\right)\cos\Omega\bar{t}. \qquad \text{(c)}$$

$D_5 = 0$ is the frequency equation for symmetric, plane-strain extensional vibrations in the radial direction. In the limit to isotropy, $\varphi_2^{(i)}$ $(i = 1,2,3)$ becomes the dilatational potential, hence the vibrations represented by (6:25) are dilatational in the isotropic state.

It is to be noticed that the symmetric plane strain frequency equation D_2 degenerates to $\tilde{D}_3 \cdot D_5$ as $n \to 0$, uncoupling the $f^{(i)}$ and $\varphi_2^{(i)}$ $(i = 1,2,3)$ potentials. For isotropy this corresponds to a disassociation of dilatational and equivoluminal motion. Furthermore, as $\zeta \to 0$, the nontorsional frequency

equation D_4 degenerates to D_5 and $\tilde{D}_1$, indicating that the axisymmetric nontorsional traveling waves emanate from the uncoupled radial-extensional and longitudinal-shear vibration fields.

Two broad distinctions in the behavior of the frequency equation are discernable. For the case of axisymmetry there are three families of cut-off frequencies which are derivable by solving the three transcendental equations $\tilde{D}_1 = 0$, $\tilde{D}_3 = 0$, and $D_5 = 0$. As ζ is increased from zero the radial-extensional and longitudinal-shear vibration fields couple and two families of branches of the frequency equation exist corresponding to the solutions of $D_3 = 0$ and $D_4 = 0$, that is, torsional and nontorsional waves. When n is not zero there are only two families of cut-off frequencies. These derive from $D_1 = 0$ and $D_2 = 0$ and correspond to longitudinal thickness-shear and plane-strain vibration fields, respectively. As ζ is increased from zero these two fields couple resulting in traveling waves for which all components of the displacement field will be generally in evidence. There is just one family of branches of the frequency equation corresponding to the solutions of equation (5:34).

5. Frequency Equation for One and Two Layered Shells

It is readily demonstrable that when two layers of the cylinder are composed of identical materials the frequency equation degenerates to a product of two determinants - one of which is the frequency equation for a two layered, traction

free cylinder. Assuming that the inner layer material constants approach the values of the center layer, the elements of the first six columns of the fourth through ninth rows of $\det|C_{ij}|$ become identical with the corresponding elements of columns 7 through 12. This permits the reduction of the frequency equation into the product of the following two determinants

$$\lim_{\substack{\delta_1^{(1)} \to \delta_1^{(2)} \\ \zeta^{(1)} \to \zeta^{(2)}}} \det|C_{ij}| = Y_1 \, F_1 = 0 \quad , \quad (i = 1,2, \quad ,6), \tag{6:26}$$

where

$$F_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{37} & C_{38} & C_{39} & C_{310} & C_{311} & C_{312} & C_{313} & C_{314} & C_{315} & C_{316} & C_{317} & C_{318} \\ C_{47} & C_{48} & C_{49} & C_{410} & C_{411} & C_{412} & C_{413} & C_{414} & C_{415} & C_{416} & C_{417} & C_{418} \\ C_{57} & C_{58} & C_{59} & C_{510} & C_{511} & C_{512} & C_{513} & C_{514} & C_{515} & C_{516} & C_{517} & C_{518} \\ C_{67} & C_{68} & C_{69} & C_{610} & C_{611} & C_{612} & C_{613} & C_{614} & C_{615} & C_{616} & C_{617} & C_{618} \\ C_{77} & C_{78} & C_{79} & C_{710} & C_{711} & C_{712} & C_{713} & C_{714} & C_{715} & C_{716} & C_{717} & C_{718} \\ C_{87} & C_{88} & C_{89} & C_{810} & C_{811} & C_{812} & C_{813} & C_{814} & C_{815} & C_{816} & C_{817} & C_{818} \\ C_{97} & C_{98} & C_{99} & C_{910} & C_{911} & C_{912} & C_{913} & C_{914} & C_{915} & C_{916} & C_{917} & C_{918} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{1013} & C_{1014} & C_{1015} & C_{1016} & C_{1017} & C_{1018} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{1113} & C_{1114} & C_{1115} & C_{1116} & C_{1117} & C_{1118} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{1213} & C_{1214} & C_{1215} & C_{1216} & C_{1217} & C_{1218} \end{vmatrix} \tag{6:27}$$

The 6X6 determinant Y_1 does not have physical significance. Furthermore, numerical analysis shows that Y_1 does not vanish.

The equation $F_1 = 0$ is the frequency equation for two layered, traction free cylinders, which are transversely

isotropic or isotropic, respectively, if equations (5:37) or (6:3) are used for the elements. For the isotropic case, Armenàkas [29] has obtained this equation in different form. The two frequency equations coalesce if F_1 is altered by replacing columns 1, 2, 7, and 8, respectively, by the difference between columns 1 and 5, 2 and 6, 7 and 11, and 8 and 12. The differences encountered between the two formulations arises from the initial choice of potentials used to separate the equations of motion.

Repeating the process just used on $\det|C_{ij}|$, but this time operating on F_1 yields

$$\lim_{\substack{\xi_1^{(1)} \to \xi_1^{(2)} \\ \xi_2^{(1)} \to \xi_2^{(2)}}} F_1 = Y_2 F_2 = 0, \quad (i = 1, 2, \ldots, 6), \tag{6:28}$$

where Y_2 is of the same nature as Y_1 and F_2 is given by

$$F_2 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{45,15} & C_{45,25} & C_{45,35} & C_{45,45} & C_{45,51} & C_{45,61} \\ C_{51,15} & C_{51,25} & C_{51,35} & C_{51,45} & C_{51,51} & C_{51,61} \\ C_{61,15} & C_{61,25} & C_{61,35} & C_{61,45} & C_{61,51} & C_{61,61} \end{vmatrix} \tag{6:29}$$

The equation $F_2 = 0$ is the frequency equation of an isotropic or anisotropic homogeneous traction free cylinder, respectively, if the elements are taken from either (5:37) or (6:3). Gazis [21] and Mirsky [39], respectively, obtained this equation for homogeneous isotropic and transversely isotropic cylinders by solving the one layer cylinder problem directly.

6. Layered Cylinders with Clamped-Free and Clamped-Clamped Boundary Conditions

In Chapter V the boundary and interface conditions utilized to obtain the frequency equation for layered cylinders with free surfaces are given in equations (5:27) and (5:28). It is a simple and straight forward problem, based on the substantial formulation of equations (5:33), to rewrite (5:27) and (5:28) to reflect any other boundary and interface conditions expressible in terms of displacements and stresses, and thence obtain directly the appropriate frequency equation to replace (5:34). It is emphasized that no additional formulation beyond the above mentioned is necessary.

The three displacement components at a clamped surface must vanish in contrast to the traction free surface requirement. Thus, recalling the construction of the frequency equation (see (5:35)), it follows that if a two layered cylinder with clamped outer surface were considered directly, a 12X12 frequency equation determinant would result. The determinant would be composed of the listed boundary and interface conditions at $\bar{r}=\bar{a}$ and $\bar{r}=\bar{b}$, and the clamped condition at $\bar{r}=\bar{c}$, which has the form

$$\begin{aligned} -u_R^{(2)} &= 0, \\ -u_\theta^{(2)} &= 0, \qquad \bar{r}=\bar{c} \\ -u_r^{(2)} &= 0 \end{aligned} \tag{6:30}$$

Similarly, if the inner surface of a two layered shell is clamped, a 12X12 determinant arises for the frequency

equation which can be considered as composed of the conditions

$$\begin{aligned} -U_r^{(a)} &= 0, \\ -U_\theta^{(a)} &= 0, \quad , \bar{r} = \bar{a} \\ -U_z^{(a)} &= 0, \end{aligned} \tag{6:31}$$

and the remaining equations of (5:35) at $\bar{r} = \bar{c}$ and $\bar{r} = \bar{d}$.

Finally, for a homogeneous cylinder clamped at both surfaces, the frequency equation is a 6X6 determinant composed only of the conditions listed in (6:30) and (6:31).

In the limit as the elastic rigidity of the inner layer approaches infinity, the frequency equation for a three layered shell with traction free surfaces approaches that of a two layered cylinder with inner surface clamped and outer surface free. Alternatively, when the rigidity of the outer layer approaches infinity, the frequency equation will approach that of a two layered shell with traction free inner surface and clamped outer surface. However, for axisymmetric ($n = 0$) motion, in the case of infinite axial wavelength, the resulting equations will include the effect of axial or rotational rigid body motion of the outer or inner rigid layer. In order to complete the clamped requirement, it is essential to secure the rigid surface by an external independent force. This is comparable to supposing that the rigid layer has infinite mass. Consequently, in those cases for which both n and ζ are zero, it will be required that both the rigidities and the density of the inner or outer layer pass to infinity.

To form the frequency equation for a two layered

cylinder with clamped inner surface, the rigidity of the inner layer of the three layered cylinder will be limited to infinity. Since a rigid material is capable of no elastic response, there is no loss of generality imposed by assuming that the inner layer is initially isotropic. The remaining layers, however, are assumed to be anisotropic. In the limit, as the stiffness of the inner layer goes to infinity, $b^{(1)}$ reduces to

$$\lim_{d_6^{(1)} \to \infty} b^{(1)} = \frac{f^{(2)}}{f^{(1)}} \lim_{d_6^{(1)} \to \infty} d_6^{(1)} \to \infty,$$

where the ratio $f^{(2)}/f^{(1)}$ remains finite. Moreover, the quantity $\eta^{(1)}$ reduces to

$$\eta_b^{(1)} = \frac{2(1-\nu^{(1)})}{(1-2\nu^{(1)})} b^{(1)} \to \infty.$$

Thus, for finite frequencies Ω, it follows from equations (6:1) that

$$\begin{aligned} \rho^{(1)2} &= \xi^{(1)2} = -\gamma^2, \\ \Omega^{(1)2} &= -\gamma^2. \end{aligned} \tag{6:32}$$

The first six columns of the 18X18 frequency determinant, composed of the elements listed in (6:3), become infinite as $d_6^{(1)} \to \infty$. To retain finiteness of the determinant when the limit is taken, these columns are each divided by $d_6^{(1)}$. Distinguishing the resulting elements of the frequency equation by $\bar{c}_{ij}$ equation (5:34) degenerates into the product of two determinants X and E_1.

$$\lim_{d_6^{(1)} \to \infty} \det |\bar{c}_{ij}| = X \cdot E_1 = 0 \tag{6:33}$$

where

$$E_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{1,10} & C_{1,11} & C_{1,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{2,10} & C_{2,11} & C_{2,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{3,10} & C_{3,11} & C_{3,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{10,1} & C_{10,2} & C_{10,3} & C_{10,10} & C_{10,11} & C_{10,12} & C_{10,13} & C_{10,14} & C_{10,15} & C_{10,16} & C_{10,17} & C_{10,18} \\ C_{11,1} & C_{11,2} & C_{11,3} & C_{11,10} & C_{11,11} & C_{11,12} & C_{11,13} & C_{11,14} & C_{11,15} & C_{11,16} & C_{11,17} & C_{11,18} \\ C_{12,1} & C_{12,2} & C_{12,3} & C_{12,10} & C_{12,11} & C_{12,12} & C_{12,13} & C_{12,14} & C_{12,15} & C_{12,16} & C_{12,17} & C_{12,18} \\ C_{13,1} & C_{13,2} & C_{13,3} & C_{13,10} & C_{13,11} & C_{13,12} & C_{13,13} & C_{13,14} & C_{13,15} & C_{13,16} & C_{13,17} & C_{13,18} \\ C_{14,1} & C_{14,2} & C_{14,3} & C_{14,10} & C_{14,11} & C_{14,12} & C_{14,13} & C_{14,14} & C_{14,15} & C_{14,16} & C_{14,17} & C_{14,18} \\ C_{15,1} & C_{15,2} & C_{15,3} & C_{15,10} & C_{15,11} & C_{15,12} & C_{15,13} & C_{15,14} & C_{15,15} & C_{15,16} & C_{15,17} & C_{15,18} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{16,13} & C_{16,14} & C_{16,15} & C_{16,16} & C_{16,17} & C_{16,18} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{17,13} & C_{17,14} & C_{17,15} & C_{17,16} & C_{17,17} & C_{17,18} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{18,13} & C_{18,14} & C_{18,15} & C_{18,16} & C_{18,17} & C_{18,18} \end{vmatrix} \qquad (6:34)$$

E_1 is the frequency equation for a two layered cylinder with clamped inner surface ($\bar{r} = \bar{b}$) and traction free outer surface ($\bar{r} = \bar{d}$).

X is a nonzero 6X6 determinant compriaed of the elements of the first six rows and columns of $\det|C_{ij}|$. Since Ω does not appear in X, this determinant is not a frequency equation.

For the case of axisymmetric traveling waves, use of equation (6:32) with n = 0 causes the frequency determinant to decompose into the product of the following three determinants, as $d\bar{\delta}^{\omega} \to \infty$:

$$\lim_{d\bar{\delta}^{\omega} \to \infty} \det|C_{ij}| = X_1 \, E_2 \, E_3 = 0, \quad n=0, \qquad (6:35)$$

where

$$E_2 = \begin{vmatrix} C_{8,11} & C_{8,12} & 0 & 0 \\ C_{11,11} & C_{11,12} & C_{11,17} & C_{11,18} \\ C_{14,11} & C_{14,12} & C_{14,17} & C_{14,18} \\ 0 & 0 & C_{17,17} & C_{17,18} \end{vmatrix}, \qquad (a)$$

and (6:36)

$$E_3 = \begin{vmatrix} C_{7,7} & C_{7,8} & C_{7,9} & C_{7,10} & 0 & 0 & 0 & 0 \\ C_{8,7} & C_{8,8} & C_{8,9} & C_{8,10} & 0 & 0 & 0 & 0 \\ C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} & C_{10,13} & C_{10,14} & C_{10,15} & C_{10,16} \\ C_{11,7} & C_{11,8} & C_{11,9} & C_{11,10} & C_{11,13} & C_{11,14} & C_{11,15} & C_{11,16} \\ C_{13,7} & C_{13,8} & C_{13,9} & C_{13,10} & C_{13,13} & C_{13,14} & C_{13,15} & C_{13,16} \\ C_{14,7} & C_{14,8} & C_{14,9} & C_{14,10} & C_{14,13} & C_{14,14} & C_{14,15} & C_{14,16} \\ 0 & 0 & 0 & 0 & C_{16,13} & C_{16,14} & C_{16,15} & C_{16,16} \\ 0 & 0 & 0 & 0 & C_{17,13} & C_{17,14} & C_{17,15} & C_{17,16} \end{vmatrix} \qquad (b)$$

Again X_1, being a function of ζ alone, does not represent a frequency determinant. The equation $E_2 = 0$ is the frequency equation for torsional waves in an infinite cylinder with clamped inner surface ($\bar{r} = \bar{b}$) and traction free outer surface ($\bar{r} = \bar{a}$). The equation $E_3 = 0$ is the frequency equation for nontorsional traveling waves in a clamped-free cylinder. From the element listing of equation (5:37) it can be seen that E_2 depends only on the potentials $f^{(i)}(r)$ ($i = 2,3$) and E_3 only on $\varphi_j^{(i)}(r)$ ($i = 2,3$; $j = 1,2$). Thus, the limit in equation (6:35) causes an uncoupling of the combined effects of the potentials.

It is readily apparent that if instead of the inner layer, the outer layer is limited to infinite rigidity, the frequency equation will again degenerate in a similar manner.

For the outer (isotropic) layer the radial wave numbers are given by equation (6:32) with 1 replaced by 3:

$$\lim_{d_3^{(3)}\to\infty}\left\{\beta^{(3)2},\beta_1^{(3)2},\beta_2^{(3)2}\right\}=-\zeta^2, \tag{6:37}$$

since

$$\lim_{d_3^{(3)}\to\infty}\beta^{(3)}\to\infty$$

Thus, proceeding directly to $\det|C_{ij}|$ with $n\neq 0$ and the two inner layers composed of transversely isotropic materials gives

$$\lim_{d_3^{(3)}\to\infty}\det|C_{ij}|=X'\cdot E_4=0, \tag{6:38}$$

with

$$E_4=\begin{vmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & 0 & 0 & 0 & 0 & 0 & 0\\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & 0 & 0 & 0 & 0 & 0 & 0\\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & 0 & 0 & 0 & 0 & 0 & 0\\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} & C_{49} & C_{4,10} & C_{4,11} & C_{4,12}\\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} & C_{57} & C_{58} & C_{59} & C_{5,10} & C_{5,11} & C_{5,12}\\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} & C_{67} & C_{68} & C_{69} & C_{6,10} & C_{6,11} & C_{6,12}\\
C_{71} & C_{72} & C_{73} & C_{74} & C_{75} & C_{76} & C_{77} & C_{78} & C_{79} & C_{7,10} & C_{7,11} & C_{7,12}\\
C_{81} & C_{82} & C_{83} & C_{84} & C_{85} & C_{86} & C_{87} & C_{88} & C_{89} & C_{8,10} & C_{8,11} & C_{8,12}\\
C_{91} & C_{92} & C_{93} & C_{94} & C_{95} & C_{96} & C_{97} & C_{98} & C_{99} & C_{9,10} & C_{9,11} & C_{9,12}\\
0 & 0 & 0 & 0 & 0 & 0 & C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} & C_{10,11} & C_{10,12}\\
0 & 0 & 0 & 0 & 0 & 0 & C_{11,7} & C_{11,8} & C_{11,9} & C_{11,10} & C_{11,11} & C_{11,12}\\
0 & 0 & 0 & 0 & 0 & 0 & C_{12,7} & C_{12,8} & C_{12,9} & C_{12,10} & C_{12,11} & C_{12,12}
\end{vmatrix} \tag{6:39}$$

The factor X' is not a frequency equation determinant, being composed of the elements of the last six rows and columns of $\det|C_{ij}|$ which are not functions of Ω. $E_4=0$ is the

frequency equation of a two layered shell with free inner surface ($\bar{r} = \bar{a}$) and clamped outer surface ($\bar{r} = \bar{c}$).

When $n = 0$, the limit gives

$$\lim_{\substack{\Omega \to \Omega_0 \\ n \to 0}} \det|\bar{C}_{ij}| = X_1' \, E_5 \, E_6 = 0, \quad n = 0, \tag{6:40}$$

where

$$E_5 = \begin{vmatrix} C_{25} & C_{26} & 0 & 0 \\ C_{55} & C_{56} & C_{5,11} & C_{5,12} \\ C_{85} & C_{86} & C_{8,11} & C_{8,12} \\ 0 & 0 & C_{11,11} & C_{11,12} \end{vmatrix}, \tag{a}$$

and (6:41)

$$E_6 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & 0 & 0 & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{47} & C_{48} & C_{49} & C_{4,10} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{67} & C_{68} & C_{69} & C_{6,10} \\ C_{71} & C_{72} & C_{73} & C_{74} & C_{77} & C_{78} & C_{79} & C_{7,10} \\ C_{91} & C_{92} & C_{93} & C_{94} & C_{97} & C_{98} & C_{99} & C_{9,10} \\ 0 & 0 & 0 & 0 & C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} \\ 0 & 0 & 0 & 0 & C_{12,7} & C_{12,8} & C_{12,9} & C_{12,10} \end{vmatrix}. \tag{b}$$

The equations $E_5 = 0$ and $E_6 = 0$ are, respectively, the frequency equations for free ($\bar{r} = \bar{a}$) - clamped ($\bar{r} = \bar{c}$) shells for the cases of torsional and nontorsional traveling waves. As before, X_1' is not a frequency equation.

If both the inner and outer layers of the shell are treated together as they have been previously on an individual basis the resulting frequency equation will be for a homogeneous transversely isotropic cylinder with clamped

$(\bar{r}=\bar{b})$ - clamped $(\bar{r}=\bar{c})$ boundary conditions. The limit gives

$$\lim_{\bar{\omega}_n^{(n)},\bar{c}_n^{(n)}\to\infty} \det\left|\bar{C}_{ij}\right| = \chi - E_7\chi^7 = 0, \tag{6:42}$$

where

$$E_7 = \begin{vmatrix} C_{11} & C_{18} & C_{19} & C_{1,10} & C_{1,11} & C_{1,12} \\ C_{81} & C_{88} & C_{89} & C_{8,10} & C_{8,11} & C_{8,12} \\ C_{91} & C_{98} & C_{99} & C_{9,10} & C_{9,11} & C_{9,12} \\ C_{10,1} & C_{10,8} & C_{10,9} & C_{10,10} & C_{10,11} & C_{10,12} \\ C_{11,1} & C_{11,8} & C_{11,9} & C_{11,10} & C_{11,11} & C_{11,12} \\ C_{12,1} & C_{12,8} & C_{12,9} & C_{12,10} & C_{12,11} & C_{12,12} \end{vmatrix}. \tag{6:43}$$

Finally, if the frequency equation (6:12) is written for the axisymmetric case, the resulting expression is

$$\lim_{\bar{\omega}_n^{(n)},\bar{c}_n^{(n)}\to\infty} \det\left|\bar{C}_{ij}\right| = \chi_1 - E_8\chi^6 - E_9\chi_1^7 = 0, \quad n=0, \tag{6:44}$$

where

$$E_8 = \begin{vmatrix} C_{11} & C_{16} \\ C_{71} & C_{76} \end{vmatrix}, \tag{a}$$

and

(6:45)

$$E_9 = \begin{vmatrix} C_{11} & C_{17} & C_{19} & C_{1,12} \\ C_{71} & C_{77} & C_{79} & C_{7,12} \\ C_{91} & C_{97} & C_{99} & C_{9,12} \\ C_{12,1} & C_{12,7} & C_{12,9} & C_{12,12} \end{vmatrix}. \tag{b}$$

$E_8 = 0$ and $E_9 = 0$ are, respectively, the torsional and nontorsional frequency equations for a clamped-clamped homogeneous cylinder.

In the limit when $\bar{\gamma}$ approaches 0 the axisymmetric $(n=0)$ frequency equations E_2, E_3, E_5, E_6, E_8, and E_9 reduce to

those of appropriately clamped two or one layered shells. It should be pointed out that in the limit as $d_6^{(i)} \to \infty$ and/or $d_6^{(3)} \to \infty$, the general frequency equation (6:21) corresponding to the axisymmetric vibrations of three layered traction free shells does not reduce to the frequency equation of the two layered clamped shell. As was indicated previously, these equations include the effect of rigid body motion of the outer or inner rigid layer. To obtain the appropriate frequency equations for clamped two and one layered shells the additional limit must be taken as $\gamma^{(i)}$ and/or $\gamma^{(3)}$ approach infinity.

7. Frequency Equations for Cylinders with Clamped-Free and Clamped-Clamped Boundary Conditions When $\gamma = n = 0$

Consideration of equations (6:23c) and (6:24c) reveals, as intimated in Section 6, that the axisymmetric cut-off modes of vibration can allow rigid body motion of a layer of the cylinder if the rigidities of that layer are indeed infinite. To obtain the appropriate frequency equations for clamped-free and clamped-clamped boundary conditions the frequency equations of Section 4 can be treated in the limit as first $d_6^{(i)}$ approaches infinity and subsequently $\gamma^{(i)}$, where i is 1 or 3, or both, depending on the choice of boundary conditions sought. It should be recognized that, regarding D_5 which is for radial extensional modes, it is not necessary to limit $\gamma^{(i)}$ of the rigid layer(s) to infinity. This follows from the fact that a rigid layer can not

extend.

Assuming that the inner (isotropic) layer is limited first, the radial wave numbers, for finite frequency and density become

$$\underset{\beta_c^{(1)} \to \infty}{\text{Limit}} \left\{ \beta_c^{(1)}, \rho_c^{(1)}, \varsigma_{ic}^{(1)} \right\} = 0 \tag{6:46}$$

Applying (6:46) to $\tilde{D}_1$, $\tilde{D}_3$, and D_5, and subsequently letting $\rho^{(1)}$ pass to infinity reduces the frequency equation to the following product

$$\underset{\rho^{(1)} \to \infty}{\text{Limit}} \left\{ \underset{\beta_c^{(1)} \to \infty}{\text{Limit}} \left\{ \tilde{D}_1 \; \tilde{D}_3 \right\} D_5 \right\} = E_{31} \; \tilde{E}_2 \; E_{32} = 0, \tag{6:47}$$

where

$$E_{31} = \begin{vmatrix} C_{9,7} & C_{9,8} & 0 & 0 \\ C_{10,7} & C_{10,8} & C_{10,13} & C_{10,14} \\ C_{15,7} & C_{15,8} & C_{15,13} & C_{15,14} \\ 0 & 0 & C_{16,13} & C_{16,14} \end{vmatrix}, \tag{a}$$

$$\tilde{E}_2 = \begin{vmatrix} C_{9,11} & C_{9,12} & 0 & 0 \\ C_{11,11} & C_{11,12} & C_{11,17} & C_{11,18} \\ C_{14,11} & C_{14,12} & C_{14,17} & C_{14,18} \\ 0 & 0 & C_{17,17} & C_{17,18} \end{vmatrix}, \tag{b} \quad (6:48)$$

$$E_{32} = \begin{vmatrix} C_{7,9} & C_{7,10} & 0 & 0 \\ C_{12,9} & C_{12,10} & C_{12,15} & C_{12,16} \\ C_{13,9} & C_{13,10} & C_{13,15} & C_{13,16} \\ 0 & 0 & C_{18,15} & C_{18,16} \end{vmatrix}. \tag{c}$$

The elements of the above frequency equations are listed in equation (6:20). Each of the three equations represents the frequency equation of a two layered cylinder clamped at $\bar{r} = \bar{b}$ and free at $\bar{r} = \bar{a}$. The appropriate

displacement field equations for E_{31}, $\tilde{E}_2$, and E_{32} are listed in equations (6:23), (6:24), and (6:25).

If the limiting process discussed above is executed on the outer layer, the following product of frequency equations is obtained.

$$\lim_{r^{(2)}\to\infty}\left\{\lim_{\bar{b}\to\infty}\left(\tilde{D}_1\tilde{D}_3\right\}D_5\right) = E_{61}\cdot\tilde{E}_5\cdot E_{62} = 0, \qquad (6:49)$$

where

$$E_{61} = \begin{vmatrix} C_{21} & C_{22} & 0 & 0 \\ C_{41} & C_{42} & C_{47} & C_{48} \\ C_{91} & C_{92} & C_{97} & C_{98} \\ 0 & 0 & C_{10,7} & C_{10,8} \end{vmatrix}, \qquad (a)$$

$$\tilde{E}_5 = \begin{vmatrix} C_{25} & C_{26} & 0 & 0 \\ C_{55} & C_{56} & C_{57} & C_{58} \\ C_{85} & C_{86} & C_{87} & C_{88} \\ 0 & 0 & C_{11,7} & C_{11,8} \end{vmatrix}, \qquad (b) \quad (6:50)$$

$$E_{62} = \begin{vmatrix} C_{13} & C_{14} & 0 & 0 \\ C_{63} & C_{64} & C_{69} & C_{6,10} \\ C_{73} & C_{74} & C_{79} & C_{7,10} \\ 0 & 0 & C_{12,9} & C_{12,10} \end{vmatrix}. \qquad (c)$$

Each of the three equations above is the frequency equation for a two layered shell clamped at $\bar{r}=\bar{c}$ and free at $\bar{r}=\bar{a}$. The elements of these determinants are listed in equation (6:20). The appropriate displacement field equations for E_{61}, $\tilde{E}_5$, and E_{62} are listed in equations (6:23), (6:24), and (6:35) with (i = 1,2).

Finally, enacting the above limits in concert produces

$$\underset{\rho^{(1)}\to\infty}{\text{Limit}}\left\{\underset{\rho_c^{(2)}\to\infty}{\text{Limit}}\left\{\widetilde{D}_1\,\widetilde{D}_3\right\}D_5\right\} = E_{91}\cdot\widetilde{E}_8\cdot E_{92} = 0\ ,\ (i=1,3), \qquad (6{:}51)$$

where

$$E_{91} = \begin{vmatrix} C_{97} & C_{98} \\ C_{10,7} & C_{10,8} \end{vmatrix},\quad \widetilde{E}_8 = \begin{vmatrix} C_{8,11} & C_{8,12} \\ C_{11,11} & C_{11,12} \end{vmatrix},\quad E_{92} = \begin{vmatrix} C_{79} & C_{7,10} \\ C_{12,9} & C_{12,10} \end{vmatrix}. \qquad (6{:}52)$$

These equations, representing the most complete reduction in complexity of the general frequency equation (5:34), are the frequency equations for homogeneous cylinders clamped at $\bar{r}=\bar{b}$ and $\bar{r}=\bar{c}$. Again equations (6:23), (6:24), and (6:25), respectively, apply to $\widetilde{E}_8$, E_{91}, and E_{92} with i restricted to the single value of 2. Baltrukonis, et al [27,78] has treated many clamped-free cases for homogeneous cylinders and rods.

8. Frequency Equation for Cylinders with Very Thin Outer Layers

An important problem in modern technology is the design and analysis of composite shells with thin facings (or outer layers). Such construction is, indeed, typical of light weight sandwich shells.

It is well known that, in the strict sense of the phrase, Bessel functions do not possess addition formulas [66], that is, it is impossible to write $Z_n(X+x)$ as an algebraic function of $Z_n(X)$ and $Z_n(x)$. There are, however, generalized formulas which pass under the name of addition formulas and which, in the case that (for specificness) $x \ll X$, can be used to provide some simplification of the frequency equation. It must be borne in mind that the

following formulations are a departure from the exact solutions presented earlier in this chapter and in Chapter V.

From a theorem due to Neumann and subsequently generalized by Graf [66], the following addition formulas for regular and modified Bessel functions of the first and second kind may be written:

$$J_\nu(\sigma)\begin{matrix}\cos\\ \sin\end{matrix}\nu\psi = \sum_{m=-\infty}^{\infty} J_{\nu+m}(X)J_m(\chi)\begin{matrix}\cos\\ \sin\end{matrix}m\theta, \qquad \text{(a)}$$

$$Y_\nu(\sigma)\begin{matrix}\cos\\ \sin\end{matrix}\nu\psi = \sum_{m=-\infty}^{\infty} Y_{\nu+m}(X)J_m(\chi)\begin{matrix}\cos\\ \sin\end{matrix}m\theta, \qquad \text{(b)}$$

$$I_\nu(\sigma)\begin{matrix}\cos\\ \sin\end{matrix}\nu\psi = \sum_{m=-\infty}^{\infty} (i)^m I_{\nu+m}(X)I_m(\chi)\begin{matrix}\cos\\ \sin\end{matrix}m\theta, \qquad \text{(c)} \qquad (6:53)$$

and

$$K_\nu(\sigma)\begin{matrix}\cos\\ \sin\end{matrix}\nu\psi = \sum_{m=-\infty}^{\infty} K_{\nu+m}(X)I_m(\chi)\begin{matrix}\cos\\ \sin\end{matrix}m\theta, \qquad \text{(d)}$$

where

$$\sigma = \left[X^2 + \chi^2 - 2X\chi\cos\theta\right]^{1/2},$$

and the numbers x and X are generally complex, ν is arbitrary, and the angle ψ is defined by

$$X - \chi\cos\theta = \sigma\cos\psi, \qquad \chi\sin\theta = \sigma\sin\psi.$$

Note that when $\theta = 0$

$$\sigma = X - \chi, \quad \psi = 0, \qquad \text{(a)}$$

and when $\theta = \pi$ (6:54)

$$\sigma = X + \chi, \quad \psi = 0 \qquad \text{(b)}$$

Using (6:54) in (6:53), with x « X and ν replaced by the integer n, produces

$$Z_n(X \pm \chi) = \left(1 \mp \frac{n\chi}{X}\right)Z_n(X) \pm \chi Z_{n-1}(X), \qquad \text{(a)} \quad (6:55)$$

and

$$W_n(X \pm 1) = \left(1 \mp \frac{n\chi}{X}\right) W_n(X) \pm \epsilon \chi W_{n-1}(X) \qquad \text{(b)} \quad \begin{matrix}(6:55)\\(\text{cont})\end{matrix}$$

In obtaining (6:55), recursion formulas (previously used) and the additional facts

$$\psi_n(X) = (-1)^n J_n(X),$$

$$I_n(x) = I_n(X),$$

were employed in conjunction with the assumptions that x is sufficiently smaller than both X and 1 so that $J_n(x)$ and $I_n(x)$ $(n=2,3,\ldots)$ are negligably small relative to $J_1(x)$ and $I_1(x)$; and $J_0(x)$, $J_1(x)$, $I_0(x)$, and $I_1(x)$ may be replaced by their equivalent small argument asymptotic forms.

If the three layered cylinder has thin outer layers (see Figure 5) then

$$\begin{aligned} a &= b - h^{(2)}, \quad h^{(2)} \ll b, \\ d &= c + h^{(3)}, \quad h^{(3)} \ll c. \end{aligned} \qquad (6:56)$$

The approximations (6:55), in light of (6:56), are valid in the low frequency, long wavelength region of the frequency spectrum and consequently the appropriate choice of Bessel functions are initially J_n and Y_n. However, since the approximation does not lose validity as χ becomes large, and since the turning point lines for the facings are steep (see equation (5:41)), the general formulations of (6:55) will be preserved so that sectors other than 7 are properly described.

Equations (5:37) must be altered to reflect (6:55).

Obviously only those elements that are written for the cylinder boundaries $\bar{a}$ and $\bar{d}$ are affected. Considering the simplest general case - torsional, axisymmetric traveling waves - the affected elements are summarized as

$$C_{25} = -\frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \left\{ \left(1 - \frac{2h^{(1)}}{b}\right) Z_2(\gamma \xi^{(1)} b) + \gamma \xi^{(1)} h^{(1)} Z_1(\gamma \xi^{(1)} b) \right\},$$

$$C_{26} = -\frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \left\{ \left(1 - \frac{2h^{(1)}}{b}\right) W_2(\gamma \xi^{(1)} b) + \epsilon_3^{(1)} \gamma \xi^{(1)} h^{(1)} W_1(\gamma \xi^{(1)} b) \right\},$$

$$C_{n,n} = -\frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \left\{ \left(1 + \frac{2h^{(1)}}{c}\right) Z_2(\gamma \xi^{(1)} c) - \gamma \xi^{(1)} h^{(1)} Z_1(\gamma \xi^{(1)} c) \right\},$$

$$C_{n,n+1} = -\frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \left\{ \left(1 + \frac{2h^{(1)}}{c}\right) W_2(\gamma \xi^{(1)} c) - \epsilon_3^{(1)} \gamma \xi^{(1)} h^{(1)} W_1(\gamma \xi^{(1)} c) \right\}.$$

Neglecting terms of the order $2h^{(i)}/b$ and $2h^{(i)}/c$ relative to 1, the above can be written as

$$\begin{aligned}
C_{25} &= C_{55} - \frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \epsilon_3^{(1)} h^{(1)} C_{55}, \\
C_{26} &= C_{56} - \frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \epsilon_3^{(1)} h^{(1)} C_{55}, \\
C_{n,n} &= C_{n,n} + \frac{1}{2} d_6^{(n)} \gamma^2 \xi^{(n)2} \epsilon_3^{(n)} h^{(n)} C_{n,n}, \\
C_{n,n+1} &= C_{n,n+1} + \frac{1}{2} d_6^{(n)} \gamma^2 \xi^{(n)2} \epsilon_3^{(n)} h^{(n)} C_{n,n+1}
\end{aligned} \tag{6:57}$$

Using (6:53) in (6:15a) provides the following equation, valid for thin outer layers.

$$\lim_{h^{(1)} \to 1} D_3 = \begin{vmatrix} C_{6,n} - Y_1 C_{8,n} & C_{5,n} - Y_1 C_{8,2} \\ C_{n,n} + Y_3 C_{n,n} & C_{n,n} + Y_3 C_{n,n} \end{vmatrix}, \tag{6:58}$$

where

$$Y_1 = \frac{1}{2} d_6^{(1)} \gamma^2 \xi^{(0)2} \epsilon_3^{(1)} h^{(1)}, \tag{a}$$

$$Y_3 = \frac{1}{2} d_6^{(n)} \gamma^2 \xi^{(n)2} \epsilon_3^{(n)} h^{(n)} \tag{b}$$

(6:59)

These results apply to shells with comparatively large relative thicknesses, for the assumptions made do not limit the thickness of the core layer. In the extreme case that $h^{(c)}$ and $h^{(s)}$ vanish, equations (6:59) vanish and (6:58) becomes exactly the frequency equation for torsional axisymmetric waves in a homogeneous, traction free cylinder.

Similar treatment of the frequency equation $D_4 = 0$ will yield corresponding approximate frequency equations for thin layered shells. However, for this case, the results are rather involved and consequently will not be discussed here.

9. N-Layered Rods and Cylindrical Shells

To generalize the theory of Chapter V to describe multi-layered shells, it is only necessary to initially suppose the cylinder to be composed of N layers, and to write additional interface conditions at each new layer interface. The free (or clamped) conditions at the inner and outer surfaces remain as previously written. Thus for an N-layered shell there will correspond a 6NX6N frequency determinant.

Any attempt to manipulate the given frequency equation (5:34) in a manner that will give way to the proper description for a three layered rod meets with only partial success. It is much more straight forward to review the boundary conditions and to observe that when the inner layer attains the geometry of a solid rod, the inner surface boundary conditions are superfluous. Furthermore, since the Bessel

functions of the second kind approach infinity as the argument approaches zero, the inner layer solutions must be modified by setting $\delta_i^{(1)}$ (i = 1,2,3) to zero. This assures that a properly obtained solution for the displacement will be finite at the origin. Thus, in writing the frequency equation for a three layered rod it is necessary to delete the first three rows and the second, fourth, and sixth columns of the cylinder frequency equation previously obtained. The extension of the rod frequency equation to N-layers is obvious and the corresponding frequency determinant will be (6N-3)X(6N-3).

The frequency equations generalized as indicated above will degenerate to lesser order determinants analogously to the degeneracies previously described for the cases of n = 0, γ = 0, and n = γ = 0.

CHAPTER VII

In Chapter VI it was shown that all branches of the frequency equation initiate, when $\zeta = 0$, with some zero or nonzero cut-off frequency. The axisymmetric cut-off frequencies are determinable by numerical analysis of the three transcendental equations given in (6:32) and the non-axisymmetric cut-off frequencies follow from the transcendental relations listed in equation (6:17). These five equations are complicated implicit relationships in Ω with parametric dependence on the elastic constants and the shell geometry. No analytical attack will be made on these equations since simplifications can only be expected when the Bessel functions are representable in asymptotic series for either very large or very small Ω. Since there is interest in the branches of the frequency equation with intermediate cut-off frequencies, a numerical analysis will be applied to obtain all cut-off frequencies sought.

When the axial wave number is not zero the branches of the frequency equation may be ascertained by solution of the two equations listed in (6:26) for axisymmetry, or equation (5:34) for non-axisymmetry. These equations involve Ω, ζ, and n implicitly, with parametric dependence on the elastic constants and shell geometry. Again a numerical solution

will be employed.

If the slope of a straight line emanating from the origin of the Ω-ζ plane is designated by m,

$$\Omega = m\zeta, \tag{7:1}$$

and this expression for Ω is used in the frequency equation, then

$$\det |C_{ij}(n, \zeta, m)| = 0 \tag{7:2}$$

represents the requirement which values of ζ must satisfy for each m and n in order for waves to propagate in the cylinder. From (7:1) the value of Ω corresponding to each value of ζ can be found. If the slope parameter m is varied in sufficiently small increments, the resulting Ω-ζ point sets become sufficiently dense on the frequency spectrum to yield the locus, for each n, of the branches of the frequency equation.

When the axial wave number is zero, the phase velocity is infinite. As ζ is increased from zero the phase velocity decreases and, on physical grounds, should approach a finite, nonzero limit when ζ approaches infinity. This follows from the fact that when the wavelength is sufficiently small, the medium in which the wave is traveling takes on the physical proportions of an infinite continuum with regard to the magnitude of the wavelength. Mindlin and Fao [20] have shown for homogeneous rods, and Armenàkas and Keck [30] for two layered rods, that all branches of the frequency equation approach asymptotically to specific values of m as ζ approaches infinity. The limiting values of m are directly

related to phase velocities of waves propagating in infinite and layered semi-infinite media. For this range of values of $\zeta(\zeta \gg 1)$ the phase and group velocities of the waves are equal, that is, the waves are nondispersed.

Along a ray of slope m the equation (7:2) will be satisfied by an infinite set of discreet values of ζ, a continuous set of values for ζ, or no ζ at all except $\zeta = 0$. If the values of ζ which satisfy (7:2) are discreet, then the branches of the frequency equation cross the ray (7:1) at the values of ζ found. If the values of ζ are continuous, the branch(s) run along the ray. If there are no values of ζ for which (7:2) is satisfied, then no branch of the frequency equation can progress to the slope m of the ray under consideration. This observation forms the basis of the asymptotic analysis of the frequency equation for $\zeta \to \infty$. The specific values of m for which the set of values of ζ changes from discreet to continuous will be found, and it will be shown that for m less than these specific values, the left side of equation (7:2) can not vanish.

Initiation of an analysis for large axial wave number is beset with a confusing array of conditions related to turning point lines and complex zones. Even the restrictions of equations (5:41), (5:42), and (5:43) do not alleviate to any great extent the diversity of conditions which must be embraced by the analysis. The frequency equation elements change from regular Bessel functions with real arguments to modified Bessel functions with real arguments

back to regular Bessel functions with complex arguments according to the range equations (5:13), (5:14), and (5:20). The formulation of the frequency equation is the same in sector 7 for all of the anisotropies under consideration. Furthermore, in sector 1 the formulation is the same for all less anisotropic and isotropic materials. However, if a layer of the cylinder is more anisotropic (either class I or II) there will be a complex zone for the radial wave numbers p and q of this layer and the frequency equation is, consequently, formulated in a different manner.

The higher sectors are of little interest in the analysis for $\gamma \to \infty$, since it will be shown that all branches of the frequency equation progress to the lowest sectors. It is convenient to proceed from sector 1 to higher sectors, showing first the conditions under which branches can exist in the lowest sectors and subsequently that the branches do indeed cross the higher sectors into the lowest when these conditions are met. Hence, the analysis will be conducted for cylinders with all layers composed of less anisotropic materials. Under this restriction the sector 1 representation of the frequency equation will be the same for the 18 cases cited in equations (5:41), (5:42), and (5:43).

1. The Frequency Equation in Sector 1 for $\gamma \to \infty$

The nine wave numbers $p^{(n)}$, $q^{(n)}$, and $s^{(n)}$ are all imaginary and, consequently, the elements of the frequency equation contain only modified Bessel functions. Since sector 1 is

bounded above by the turning point line whose slope is the least of $b^{(i)}(i=1,2,3)$ (Figure 6), any ray used in this sector will have slope m which is less than the least $b^{(i)}(i=1,2,3)$. Under this restriction it follows that

$$\underset{\gamma\to\infty}{\text{Limit}}\left\{\xi^{(1)},\xi^{(2)},\xi^{(3)}\right\}\to\infty \tag{7:3}$$

Because of (7:3) it follows that, with $\gamma=\gamma$ wave number,

$$\underset{\gamma\to\infty}{\text{Limit}}\left\{Z_n(\gamma r),W_n(\gamma r)\right\}=\frac{e^{\gamma r}}{(2\pi\gamma r)^{1/2}}\left(\frac{\pi}{2\gamma r}\right)^{1/2}e^{-\gamma r}\,, \tag{7:4}$$

and due to the construction of the frequency equation determinant, the factors raised to the $\frac{1}{2}$ power in (7:4) can, and henceforth will, be ignored (they factor out of the determinant). Introducing (7:3) and (7:4) into equations (5:37), and retaining only terms with bounded factors (excluding the exponential factors), the elements of the frequency equation take the form

$$\begin{aligned}
C_{11}&=\frac{-d_4^{(1)}\bar{a}^2}{d_2^{(1)}+d_4^{(1)}}\left[-d_1^{(1)}e^{\xi^{(1)}2}+d_5^{(1)}\frac{m^2}{\xi^{(1)}2}-d_5^{(1)}\right]e^{\xi^{(1)}\bar{a}}\,,\\
C_{12}&=\frac{-d_4^{(2)}\bar{a}^2}{d_2^{(2)}+d_4^{(2)}}\left[-d_1^{(2)}e^{\xi^{(2)}2}+d_5^{(2)}\frac{m^2}{\xi^{(2)}2}-d_5^{(2)}\right]e^{\xi^{(2)}\bar{a}}\,,\\
C_{13}&=-d_4^{(3)}\bar{a}^2\left[-d_1^{(3)}e^{\xi^{(3)}2}+d_5^{(3)}\frac{m^2}{\xi^{(3)}2}-d_5^{(3)}\right]e^{\xi^{(3)}\bar{a}}\,,\\
C_{14}&=-d_4^{(4)}\bar{a}^2\left[-d_1^{(4)}e^{\xi^{(4)}2}+d_5^{(4)}\frac{m^2}{\xi^{(4)}2}-d_5^{(4)}\right]e^{\xi^{(4)}\bar{a}}\,,\\
C_{15}&=C_{16}=0\,,\\
C_{21}&=C_{22}=C_{23}=C_{24}=0\,,\\
C_{25}&=-\frac{1}{2}d_6^{(1)}\xi^{(1)2}\bar{a}^2e^{\xi^{(1)}\bar{a}}\,,\\
C_{26}&=-\frac{1}{2}d_6^{(2)}\xi^{(2)2}\bar{a}^2e^{\xi^{(2)}\bar{a}}\,,
\end{aligned} \tag{7:5}$$

$$C_{31} = \frac{d_4^{(1)}}{d_3^{(1)}+d_4^{(1)}}\left[-d_1^{(1)}\varepsilon^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} - d_3^{(1)}\right]\varepsilon^{(1)}\bar{a}^2\, e^{\mathrm{i}k^{(1)}\bar{a}},$$

$$C_{32} = \frac{-d_3^{(1)}}{d_3^{(1)}+d_4^{(1)}}\left[-d_1^{(1)}\varepsilon^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} - d_3^{(1)}\right]\varepsilon^{(1)}\bar{a}^2\, e^{-\mathrm{i}k^{(1)}\bar{a}},$$

$$C_{33} = d_4^{(1)}\left[-d_1^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} - d_3^{(1)}\right]\beta^{(1)}\bar{a}^2\, e^{\mathrm{i}k^{(1)}\bar{a}},$$

$$C_{34} = -d_4^{(1)}\left[-d_1^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} - d_3^{(1)}\right]\beta^{(1)}\bar{a}^2\, e^{-\mathrm{i}k^{(1)}\bar{a}},$$

$$C_{35} = C_{36} = 0,$$

$$C_{71} = \varepsilon^{(1)}\, e^{\mathrm{i}k^{(1)}b},$$

$$C_{72} = -\varepsilon^{(1)}\, e^{-\mathrm{i}k^{(1)}b},$$

$$C_{73} = (d_3^{(1)}+d_4^{(1)})\beta^{(1)}\, e^{\mathrm{i}k^{(1)}b},$$

$$C_{74} = -(d_3^{(1)}+d_4^{(1)})\beta^{(1)}\, e^{-\mathrm{i}k^{(1)}b},$$

$$C_{75} = C_{76} = 0,$$

$$C_{81} = C_{82} = C_{83} - C_{84} = 0,$$

$$C_{85} = -\varepsilon^{(1)}\, e^{\mathrm{i}k^{(1)}b},$$

$$C_{86} = \varepsilon^{(1)}\, e^{-\mathrm{i}k^{(1)}b},$$

$$C_{91} = \frac{b}{d_3^{(1)}+d_4^{(1)}}\left[-d_1^{(1)}\varepsilon^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} + d_4^{(1)}\right]e^{\mathrm{i}k^{(1)}b},$$

$$C_{92} = \frac{b}{d_3^{(1)}+d_4^{(1)}}\left[-d_1^{(1)}\varepsilon^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} + d_4^{(1)}\right]e^{-\mathrm{i}k^{(1)}b},$$

$$C_{93} = b\left[-d_1^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} + d_4^{(1)}\right]e^{\mathrm{i}k^{(1)}b},$$

$$C_{94} = b\left[-d_1^{(1)}\beta^{(1)2} - d_4^{(1)}\frac{m^2}{k^{(1)2}} + d_4^{(1)}\right]e^{-\mathrm{i}k^{(1)}b},$$

$$C_{95} = C_{96} = 0$$

(7:5) (cont)

The remaining elements of the frequency equation may be formed from the above by use of the instructions in (5:37).

It is significant to notice that equations (7:5) and (6:13) have exactly the same nonzero elements. Thus, it follows that the above elements will yield a product of two determinants as in the case for n = 0. Further, since n does not occur in equations (7:5), the forthcoming results will be valid for all n.

Using (7:5) in (5:34) gives

$$\lim_{\xi \to \infty} \det |c_{ij}(n, \xi, m)| = D_3 \cdot D_4 = 0, \tag{7:6}$$

where D_3 and D_4 are arranged from the elements listed in (6:25).

Considering first the possibility that $D_3 = 0$ yields

$$\left(d_6^{(2)} \xi^{(2)} + d_6^{(3)} \xi^{(3)}\right)\left(d_6^{(2)} \xi^{(2)} + d_6^{(3)} \xi^{(3)}\right) = 0. \tag{7:7}$$

The first term of (7:7), incorporating (5:20), gives

$$\sqrt{d_4^{(2)} \xi_6^{(2)} \left(1 - m^2/\xi^{(1)2}\right)} = -\sqrt{d_6^{(3)} \xi_6^{(3)} \left(1 - m^2/\xi^{(3)2}\right)}.$$

In sector 1 this equality can never be realized. The same is true for the second factor of (7:7). Consequently, regardless of the type of shell under consideration, and whether n is zero or not, no branch of the frequency equation for torsional motion (or coming strictly from the potential set $f^{(i)}$ (i = 1,2,3)) ever penetrates into sector 1.

Turning to D_4, using (6:15b) and (7:5), and retaining only the largest terms, D_4 degenerates to

$$D_4 = D_6^{(2)} \cdot D_7^{(1,2)} \cdot D_7^{(3,2)} \cdot D_6^{(3)} = 0, \tag{7:8}$$

where

$$D_6^{(1)} = \begin{vmatrix} C_{12} & C_{14} \\ C_{32} & C_{34} \end{vmatrix}, \qquad (a)$$

$$D_7^{(1,2)} = \begin{vmatrix} C_{41} & C_{43} & C_{46} & C_{4,10} \\ C_{61} & C_{63} & C_{66} & C_{6,10} \\ C_{71} & C_{73} & C_{76} & C_{7,10} \\ C_{91} & C_{93} & C_{96} & C_{9,10} \end{vmatrix}, \qquad (b)$$

$$D_7^{(1,3)} = \begin{vmatrix} C_{10,7} & C_{10,9} & C_{10,10} & C_{10,12} \\ C_{12,7} & C_{12,9} & C_{12,10} & C_{12,12} \\ C_{13,7} & C_{13,9} & C_{13,10} & C_{13,12} \\ C_{15,7} & C_{15,9} & C_{15,10} & C_{15,12} \end{vmatrix}, \qquad (c)$$

$$D_6^{(3)} = \begin{vmatrix} C_{10,13} & C_{10,15} \\ C_{12,13} & C_{12,15} \end{vmatrix}. \qquad (d)$$

(7:9)

The determinants $D_6^{(i)}$ $(i=1,3)$ represent the same frequency equation but at the free surfaces $\bar{r}=\bar{a}$ and $\bar{r}=\bar{d}$, respectively. Similarly, the two determinants $D_7^{(i,2)}$ $(i=1,3)$ are the same except that $D_7^{(1,2)}$ is for $\bar{r}=\bar{b}$ and $D_7^{(3,2)}$ for $\bar{r}=\bar{c}$. These equations will be discussed separately.

A. The Equation $D_6^{(i)}=0$ $(i=1,3)$

For isotropic conditions $D_6^{(i)}=0$ becomes

$$D_6^{(i)} = \begin{vmatrix} \left(2-\frac{m^2}{\bar{s}^{(i)2}}\right) & -2\bar{s}^{(i)} \\ 2\bar{s}^{(i)} & -\left(2-\frac{m^2}{\bar{s}^{(i)2}}\right) \end{vmatrix}, \quad (i=1,3) \qquad (7:10)$$

This is the frequency equation for Rayleigh surface waves [47,79].

If $D_6^{(i)}=0$ is satisfied, only the integration constants

$\bar{R}_1^{(i)}$ and $\bar{R}_2^{(i)}$ are nonzero. Furthermore, since

$$\frac{dK_n(\gamma\bar{r})}{d\bar{r}} \sim -\gamma K_{n+1}(\gamma\bar{r}) \quad \text{as} \quad \gamma \to \infty,$$

the displacement field equations (5:24), for $\zeta \to \infty$, contain only $u_r^{(i)}$ and $u_z^{(i)}$ components. This is directly comparable to the case for $n = 0$ and was to be expected since the asymtotic behavior in sector 1 is independent of n.

It is easy to show that the displacement field decays from the $\bar{r} = \bar{a}$ free surface of the cylinder in the case of $D_6^{(i)} = 0$, or from the free surface $\bar{r} = \bar{d}$ if $D_6^{(i)} = 0$. The nature of the asymptotic expansion and the relative size of ζ is obtained by recalling that the Rayleigh surface waves satisfy the boundary equations and equations of motion written for a single material half space. Thus ζ must be sufficiently large so that in the radial direction the wave motion decays completely within the thickness of the inner or outer layer. The displacement field will approximate the surface wave phenomena for finite, large ζ - being a more satisfactory approximation for a set value of ζ in thicker layers than thin.

Expanding (7:10) gives

$$\left(2 - \frac{m^2}{\psi^{(i)2}}\right)^2 = 4\sqrt{1 - m^2/b^{(i)2}}\sqrt{1 - m^2/\psi^{(i)2}}$$

But

$$b^{(i)} = \frac{c_1^{(i)}}{c_4^{(i)}}\psi^{(i)},$$

$$\frac{c_1^{(i)}}{c_4^{(i)}} = \frac{1 - 2\nu^{(i)}}{2 - 2\nu^{(i)}}$$

Then

$$\left(2-\frac{m^2}{k^{(i)2}}\right)^2 = 4\sqrt{1-\frac{m^2}{k^{(i)2}}}\sqrt{1-\frac{1-2\nu^{(i)}}{2-2\nu^{(i)}}\frac{m^2}{k^{(i)2}}} \qquad (7:11)$$

In this form it is clear that the value of the real roots of the Rayleigh equation are dependent on Poisson's ratio. It can be shown [80] that

$$0.87 < \frac{v}{\beta^{(i)}} = m_R^{(i)} < 0.96 \qquad (a)\ (7:12)$$

as $\nu^{(i)}$ varies from 0 to $\frac{1}{2}$. It follows that if the ray with slope $m_R^{(i)}b^{(i)}$, bounded by the limits

$$0.87\,b^{(i)} < m_R^{(i)}b^{(i)} < 0.96\,b^{(i)}, \quad (i=1,3), \qquad (b)\ (7:12)$$

lies in sector i when $\xi \gg i$, the i-th (i = 1,3) layer of the cylinder has a displacement field which corresponds to a Rayleigh surface wave, and an undispersed branch of the frequency equation lies on the locus

$$\Omega = m_R^{(i)}k^{(i)}\xi, \quad (i=1,3).$$

Notice that it is not always possible for this locus to be in sector i. Indeed, if $0.87b^{(i)}$ (i = 1,3) is greater than the sector i boundary slope, $m_R^{(i)}b^{(i)}$ (i = 1,3) can never lie in sector 1. This restriction does not preclude the existence of Rayleigh waves in the inner or outer layers, it merely indicates that the mechanism involved in the appearance of surface waves is not so simple when the shell is fabricated of materials placing it in the sandwich shell class (5:41).

Since in the isotropic case - and especially for core layered shells - it is possible for $D_6^{(i)} = 0$ to have a solution

in sector 1, it is reasonable to expect that there will also be an analogous solution for at least some range of anisotropic materials. In particular, for less anisotropic materials, it is reasonable to expect a solution. In every term of the frequency equation the variables Ω and χ appear only as the quotient Z, hence, if a solution exists in sector 1, regardless of the degree of anisotropy, it necessarily represents an undispersed mode of wave motion.

Very little can be discovered qualitatively for an equation even as simple as (7:10a) when all the terms are properly represented from equation (7:5). Since the radial wave numbers appear as squares and first powers, little can be accomplished without extensive rationalization - a process which by its intrinsic character destroys the very information being sought; sector 1 values of m which cause (7:10a) to vanish. Numerical analysis will have to be employed for the anisotropic case.

B. The Equation $D_7^{(i1)} = 0$ $(i = 1,3)$

As was done with the Rayleigh surface wave equation, the discussion will commence with an investigation of the isotropic case. To avoid needless notation the superscript (i1) will be replaced by (1) and it will be understood that similar results hold if (1) is replaced by (3). The equation at either interface $\bar{r} = \bar{b}$ or $\bar{r} = \bar{c}$ is identical, the only difference being in the set of integration constants which must be non-trivial.

For isotropy $D_7^{(3)}$ becomes

$$D_7^{(3)} = \begin{vmatrix} 2d_6^{(1)}\beta^{(1)} & -d_6^{(1)}(1+\varepsilon^{(1)}) & 2d_6^{(2)}\beta^{(2)} & d_6^{(2)}(1+\varepsilon^{(2)}) \\ -d_6^{(1)}(1+\varepsilon^{(1)}) & 2d_6^{(1)}\varepsilon^{(1)} & d_6^{(2)}(1+\varepsilon^{(2)}) & 2d_6^{(2)}\varepsilon^{(2)} \\ \beta^{(1)} & -1 & \beta^{(2)} & 1 \\ 1 & -\varepsilon^{(1)} & -1 & -\varepsilon^{(2)} \end{vmatrix} . \qquad (7:13)$$

Equation (7:13) is known as the Stoneley wave equation. Stoneley introduced it in 1924 [81] and indicated, rather summarily, that roots exist for a restricted range of values of the material constants. Since then Scholte [82-87], Koppe [88], and Sezawa and Kanai [89-92] have discussed it extensively. A discussion of the extent of variation of the material constants for which (7:13) may possess a solution in sector 1 will be presented now based on the above references and the author's work [30].

For convenience, the following parameter will be defined

$$\bar{\mu} = \frac{d_6^{(1)}}{d_6^{(2)}} = \frac{c_{44}^{(1)}}{c_{44}^{(2)}} \quad , (i = 1,3) \qquad (7:14)$$

No ambiguity will result from omitting a superscript on $\bar{\mu}$ since in either $D_7^{(3)}$ or $D_7^{(3,2)}$ only one of the $d_6^{(i)}$ $(i = 1,3)$ appears.

To proceed with the discussion a coordinate plane with ordinate axis $\bar{\mu}$ and abscissa $b^{(2)}$ will be employed. Rearranging (7:13) to reflect definition (7:14) yields

$$D_7^{(3)} = \begin{vmatrix} 2\bar{\mu}\beta^{(1)} & -\bar{\mu}(1+\varepsilon^{(1)}) & 2\beta^{(2)} & (1+\varepsilon^{(2)})^2 \\ -\mu(1+\varepsilon^{(1)}) & 2\bar{\mu}\varepsilon^{(1)} & (1+\varepsilon^{(2)}) & 2\varepsilon^{(2)} \\ \beta^{(1)} & -1 & \beta^{(2)} & 1 \\ 1 & -\varepsilon^{(1)} & -1 & -\varepsilon^{(2)} \end{vmatrix} = 0 \qquad (7:15)$$

The intent will be to plot, on a $\bar{\mu}$-$b^{(2)}$ plane, the boundary curves which determine the region within which (7:15) can be satisfied for some choice of m in sector 1 of the $\bar{\mu}$-$\bar{\gamma}$ plane.

If $b^{(1)}$ is the turning point line with least slope, then the slope of any ray considered must be less than $b^{(1)}$. But, $b^{(1)} \equiv 1$ by definition. If, on the other hand, $b^{(1)}$ is less than $b^{(2)}$, then no ray may be considered which has slope greater than $b^{(1)}$. In summary, one notes that the permissable range of values for the ray slope m is

$$0 < m < 1 \qquad \text{if} \quad b^{(1)} > b^{(2)}, \tag{a}$$

or (7:16)

$$0 < m < b^{(1)} \qquad \text{if} \quad b^{(2)} > b^{(1)} \tag{b}$$

The positive $b^{(2)}$ axis encompasses all possible variations of inequalities (7:16).

Expanding (7:15), yields

$$0 = \bar{\mu}^2\left(1-\alpha^{(1)}\beta^{(2)}\right)\left[4\alpha^{(1)}\beta^{(1)} - (1+\alpha^{(1)2})^2\right] - \bar{\mu}\left\{(1-\alpha^{(1)}\beta^{(2)})\left[4\alpha^{(1)}\beta^{(1)} - (1+\alpha^{(1)2})^2\right] + \right.$$

$$+ \frac{m^4}{b^{(2)4}}\left(1-\frac{b^{(1)2}}{b^{(2)2}}\right)^2 - \frac{m^4}{b^{(2)4}}\left(\beta^{(1)} + \frac{b^{(2)2}}{b^{(1)2}}\beta^{(2)}\right)\left(\alpha^{(1)} + \frac{b^{(1)2}}{b^{(2)2}}\alpha^{(2)}\right) +$$

$$\left. + (1-\alpha^{(1)}\beta^{(2)})\left[4\alpha^{(2)}\beta^{(2)} - (1+\alpha^{(2)2})^2\right]\right\} + (1-\alpha^{(1)}\beta^{(2)})\left[4\alpha^{(1)}\beta^{(2)} - (1+\alpha^{(1)2})^2\right].$$

This form of the Stonaley frequency equation was first presented by Keck and Armenàkas [30] and is conducive to investigation on the $\bar{\mu}$-$b^{(2)}$ plane. Expanding the $\alpha^{(i)}$ and $\beta^{(i)}$ yields

$$\bar{\mu}^2\left(1-\sqrt{1-m^2/b^{(2)2}}\sqrt{1-m^2/\eta^{(2)2}}\right)\left[4\sqrt{1-m^2/b^{(2)2}}\sqrt{1-m^2/\eta^{(2)2}} - (2-m^2/\eta^{(2)2})^2\right] -$$

$$- \bar{\mu}\left\{\left(1-\sqrt{1-m^2/b^{(2)2}}\sqrt{1-m^2/\eta^{(2)2}}\right)\left[4\sqrt{1-m^2/b^{(2)2}}\sqrt{1-m^2/\eta^{(2)2}} - (2-m^2/\eta^{(2)2})^2\right] + (\text{cont.})\right. \tag{7:17}$$

$$(\text{cont}) + \frac{m^4}{\bar{\mu}^{(12)}}\left(1-\frac{\bar{\mu}^{(12)}}{\bar{\mu}^{(21)}}\right)^2 - \frac{m^4}{\bar{\mu}^{(21)}}\left(\sqrt{1-m^2/\bar{\mu}^{(12)}} + \frac{\bar{\mu}^{(12)}}{\bar{\mu}^{(21)}}\sqrt{1-m^2/\bar{\mu}^{(12)}}\right)\left(\sqrt{1-m^2/\bar{\mu}^{(12)}} + \frac{\bar{\mu}^{(12)}}{\bar{\mu}^{(21)}}\sqrt{1-m^2/\bar{\mu}^{(12)}}\right) +$$

$$+\left(1-\sqrt{1-m^2/\bar{\mu}^{(12)}}\sqrt{1-m^2/\bar{\mu}^{(21)}}\right)\left[4\sqrt{1-m^2/\bar{\mu}^{(12)}}\sqrt{1-m^2/\bar{\mu}^{(21)}} - (2-m^2/\bar{\mu}^{(12)})^2\right] + \qquad (7:17)$$

$$+\left(1-\sqrt{1-m^2/\bar{\mu}^{(12)}}\sqrt{1-m^2/\bar{\mu}^{(21)}}\right)\left[4\sqrt{1-m^2/\bar{\mu}^{(21)}}\sqrt{1-m^2/\bar{\mu}^{(12)}} - (2-m^2/\bar{\mu}^{(21)})^2\right] = 0 \qquad (\text{cont})$$

This equation displays the complete parametric dependence of $\bar{\mu}$ on m and $b^{(12)}$.

When $\bar{\mu} = 0$, (7:17) reduces to the Rayleigh equation for material 2. The only nontrivial solution for (7:17) is, then,

$$m = \Omega/\zeta = m_R^{(2)} \quad , (\bar{\mu} = 0) \; . \qquad (7:18)$$

On the phase plane, nondimensionalized relative to $v_s^{(2)}$, the ray $\Omega = m_R^{(2)}\zeta$ is stationary, that is, $\Omega = m_R^{(2)}\zeta$ always lies below the ray $\Omega = \zeta$ by an amount bracketted by the limits given in (7:12). Thus, for $\bar{\mu} = 0$, $D_7^{(12)} = 0$ only if (7:18) holds, and this value of m is insector 1 of the Ω-ζ plane only if $b^{(2)} \geq m_R^{(2)}$. When $b^{(12)} = m_R^{(2)}$ the locus of the branch of the frequency equation and the lowest turning point line on the Ω-ζ plane are coincident, and as $b^{(12)}$ increases, the turning point line rises above the branch of the frequency equation given by $\Omega = m_R^{(2)}\zeta$. On the $\bar{\mu}$-$b^{(12)}$ plane the $b^{(12)}$ axis, for $b^{(12)} \geq m_R^{(2)}$ may be considered the locus for the solution of $D_7^{(12)} = 0$, which is satisfied by $m = m_R^{(2)}$.

When $\bar{\mu} = \infty$, $D_7^{(12)} = 0$ degenerates to $D_6^{(1)} = 0$, which possesses a nontrivial solution

$$m = \frac{\Omega}{\zeta} = m_R^{(1)}b^{(1)} \quad , (\bar{\mu} \gg 1) \; . \qquad (7:19)$$

Notice that $m_R^{(1)}b^{(1)}$ is always less than $b^{(1)}$ by an amount given by (7:12). Consequently, so long as $m_R^{(1)}b^{(1)}$ is less than or equal to 1, a branch of the frequency equation, given by the

equation $\Omega = m_R^{(1)} b^{(1)} \zeta$, lies in sector 1 of the Ω-ζ plane. On the $\bar{\mu}$-$b^{(1)2}$ plane, at $\bar{\mu} = \infty$, the line parallel to the $b^{(1)2}$ axis may be thought of as the locus for the solution of $D_7^{(1)2} = 0$, which is satisfied by $m = m_R^{(1)} b^{(1)}$. This line starts at $b^{(1)2} = 0$ and continues past $b^{(1)2} = 1$ to $b^{(1)2} = 1/m_R^{(1)2}$ which is the upper limit of $b^{(1)2}$ for which the locus $m_R^{(1)} b^{(1)} \zeta$ remains in sector 1 of the Ω-ζ plane. At this upper extent the lowest turning point line and the lowest branch of the frequency equation coincide.

The solutions $m = m_R^{(1)}$ (for $\bar{\mu} = 0$) and $m = m_R^{(1)} b^{(1)}$ (for $\bar{\mu} \to \infty$) are the lowest values of m in sector 1 for which the equation $D_7^{(1,2)} = 0$ can be satisfied. For $b^{(1)} \leq 1$, the maximum value of m which lies in sector 1 is just less than $b^{(1)}$ itself. Thus, if $m = b^{(1)}$ is substituted into (7:17) the resulting equation, when plotted on the $\bar{\mu}$-$b^{(1)2}$ plane, will bound the region in which $D_7^{(1,2)} = 0$ is satisfiable by m in sector 1. Performing this substitution, and noting that $b^{(2)} = 1$, equation (7:17) gives

$$-\bar{\mu}^2 \left[1 - \sqrt{1 - b^{(1)2}} \sqrt{1 - b^{(1)2}/m_R^{(1)2}} \right] + \bar{\mu} \left[2(2 - b^{(1)2}) + 2 b^{(1)2} \sqrt{1 - b^{(1)2}} \sqrt{1 - b^{(1)2}/m_R^{(1)2}} - 4 \sqrt{1 - b^{(1)2}} \sqrt{1 - b^{(1)2}/m_R^{(1)2}} \right] - \left[(2 - b^{(1)2})^2 - 4 \sqrt{1 - b^{(1)2}} \sqrt{1 - b^{(1)2}/m_R^{(1)2}} \right] = 0. \qquad (7:20)$$

When $b^{(1)2} = 1$, equation (7:20) reduces to

$$\bar{\mu}^2 - 2\bar{\mu} + 1 = 0$$

Thus, the quadratic (7:20) possesses a double root $\bar{\mu} = 1$ at $b^{(1)2} = 1$. As $b^{(1)2}$ decreases, the two root values of $\bar{\mu}$ separate. When $b^{(1)2} = m_R^{(1)2}$ the lower root of (7:20) is $\bar{\mu} = 0$. For $b^{(1)2} < m_R^{(1)2}$ this root becomes negative, which is unacceptable on physical grounds. The upper root alone is significant for $b^{(1)2}$ less

than $m_R^{(12)}$. As $b^{(12)} \to 0$, (7:20) becomes

$$-\bar{\mu}^2\left[1-(1-\tfrac{1}{2}b^{(12)})(1-\tfrac{1}{2}b^{(12)}/\eta^{(12)})\right]+\bar{\mu}\left[4-2b^{(12)}+b^{(12)}\times\right.$$
$$\left.\times\left(1-\tfrac{1}{2}b^{(12)}\right)\sqrt{1-b^{(12)}/\eta^{(12)}}-4\left(1-\tfrac{1}{2}b^{(12)}\right)\left(1-\tfrac{1}{2}b^{(12)}/\eta^{(12)}\right)\right]-$$
$$-\left[4-4b^{(12)}-4\left(1-\tfrac{1}{2}b^{(12)}\right)\left(1-\tfrac{1}{2}b^{(12)}/\eta^{(12)}\right)\right]=0,$$

which is a quadratic expression in $\bar{\mu}$ with positive root

$$\bar{\mu}=\frac{2/\eta^{(12)}+\sqrt{1-b^{(12)}/\eta^{(12)}}+\sqrt{5-b^{(12)}/\eta^{(12)}+(4/\eta^{(12)})\sqrt{1-b^{(12)}/\eta^{(12)}}}}{(1+1/\eta^{(12)})} \qquad (7:21)$$

Summarizing the above discussion, the region on the $\bar{\mu}$-$b^{(12)}$ plane for which $D_7^{(12)}=0$, for any value of m in sector 1 of the Ω-γ plane, is bounded by the curves: equation (7:20), $\bar{\mu}=0$, and $b^{(12)}=1$. This region is valid only when the lowest turning point line is $Z=b^{(1)}$, which, in turn, is the lowest only for $b^{(0)} \le b^{(1)}=1$. In the region where $D_7^{(12)}=0$, the value of m (for any permissable choice of $\bar{\mu}$ and $b^{(12)}$) is greater than $m_R^{(1)}$ if $\bar{\mu} \le 1$ and greater than $m_R^{(1)}b^{(1)}$ if $\bar{\mu} \ge 1$. Figure 7 illustrates these results.

The discussion for the alternate case, $b^{(12)} \ge 1$, goes rapidly. The $b^{(12)}$ axis is a boundary curve since along this axis ($\bar{\mu}=0$) $D_7^{(12)}=0$ is always satisfied by $m_R^{(1)}$ which, in turn, always lies in sector 1 when $b^{(12)}$ is the lowest turning point line. When $\bar{\mu} \to \infty$, the line parallel to the $b^{(12)}$ axis extending from $b^{(12)}=1$ to $b^{(12)}=1/m_R^{(12)}$ is an upper boundary since $D_7^{(12)}=0$ is satisfied if $m=m_R^{(1)}b^{-1}$, and this choice of m lies in sector 1 of the Ω-γ plane so long as $b^{(12)} \le 1/m_R^{(12)}$. The curve $b^{(12)}=1$ is a third boundary of the region being sought.

The remaining boundary of the region in question is obtained from equation (7:17) by using the substitution $m=1$.

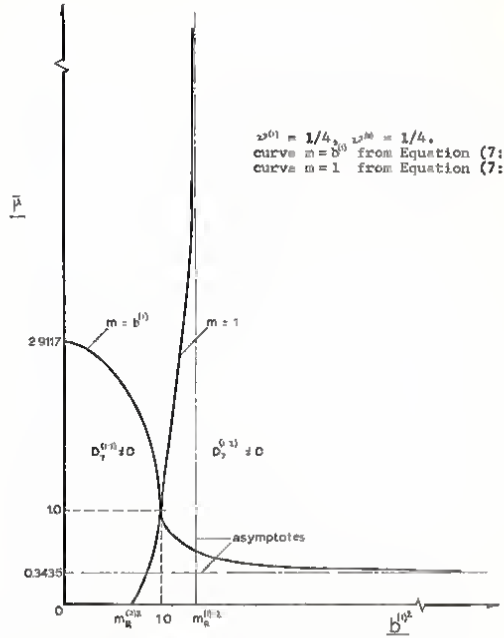

FIGURE 7. Illustration of the region on the $\bar{\mu}$-$b^{(1)}$ plane where values of $\bar{\mu}$, $b^{(1)}$ and m allow $D_7^{(1,1)} = 0$ in sector 1 of the frequency spectrum.

This follows from the fact that so long as $m \in 1$, equation (7:17), if satisfied, is satisfied by a value of m in sector 1 of the Ω-ζ plane. Using this value of m, and the fact that $\bar{b}^{(2)} = i$, (7:17) yields

$$-\bar{\mu}^2\left[(2-1/\bar{b}^{(12)})^2 - 4\sqrt{1-1/\bar{b}^{(12)}}\sqrt{1-1/\eta_E^{(12)}}\right] + \bar{\mu}\left[2(2-1/\bar{b}^{(12)}) + 4(1/\bar{b}^{(12)})\sqrt{1-1/\eta_E^{(12)}}\sqrt{1-1/\bar{b}^{(12)}} - 4\sqrt{1-1/\bar{b}^{(12)}}\sqrt{1-1/\eta_E^{(12)}}\right] - \left[1-\sqrt{1-1/\bar{b}^{(12)}}\sqrt{1-1/\eta_E^{(12)}}\right] = 0. \qquad (7:22)$$

Equation (7:22) possesses a double root, $\bar{\mu} = i$, at $\bar{b}^{(12)} = i$. When $\bar{b}^{(12)}$ equals $1/m_E^{(12)}$ one root of (7:22) is $\mu \to \infty$, the other is less than 1. For $\bar{b}^{(12)} > 1/m_E^{(12)}$, only the lower root retains its significance. When $\bar{b}^{(12)} \to \infty$ equation (7:22) yields

$$-\bar{\mu}\left[4 - 4/\bar{b}^{(12)} - 4(1-1/2\bar{b}^{(12)})(1-1/2\eta_E^{(12)})\right] + \bar{\mu}\left[4 - 2/\bar{b}^{(12)} + 4(1/\bar{b}^{(12)})\sqrt{1-1/\eta_E^{(12)}}\,(1-1/2\bar{b}^{(12)}) - 4(1-1/2\bar{b}^{(12)})(1-1/2\eta_E^{(12)})\right] + \left[1-(1-1/2\bar{b}^{(12)})(1-1/2\eta_E^{(12)})\right] = 0,$$

which gives the asymptote for (7:22) when $\bar{b}^{(12)} \to \infty$:

$$\mu = \frac{-\sqrt{1-1/\eta_E^{(12)}} - 2\bar{b}^{(12)}/\eta_E^{(12)} + \sqrt{5-1/\eta_E^{(12)} + 4(\bar{b}^{(12)}/\eta_E^{(12)})\sqrt{1-1/\eta_E^{(12)}}}}{4(1-\bar{b}^{(12)}/\eta_E^{(12)})}. \qquad (7:23)$$

The description of the partition of the $\bar{\mu}$-$\bar{b}^{(12)}$ plane is complete and Figure 7 shows the region of variables for which $D_7^{(2)} = 0$ can be satisfied by a ray with slope m lying in sector 1.

Two comments concerning Figure 7 are in order. First, the point $\bar{\mu} = i$, $\bar{b}^{(12)} = i$ corresponds to the case for single material. Equation (7:17) vanishes identically, but in essence would never have appeared had the two layers been identical and also perfectly bonded. Second, the line

$b^{(2)} = 1$ corresponds to Wiechert's condition for the possibility of existence of a solution to $D_7^{(2)} = 0$. The requirement $b^{(2)} = 1$ is identical to $\mu^{(1)}/\mu^{(2)} = \rho^{(1)}/\rho^{(2)}$. This is the condition which Stoneley [81] implied must be nearly satisfied for a solution of the determinant to exist. Wiechert's condition was developed as a requirement to satisfy certain analytical models for earthquake phenomena in layered media. From Figure 7 it is clear that conditions very divergent from the Wiechert requirement lead to a solution of $D_7^{(2)} = 0$.

When a degree of anisotropy is introduced, all discussion must be based on equation (7:9). It is again reasonable to suppose that less anisotropic media have a material dependence somewhat akin to the isotropic case, and that the anisotropic version of $D_7^{(2)}$ possesses a sector 1 solution.

C. Summary of Sector 1 Limits

For sandwich shells both $b^{(1)}$ and $b^{(3)}$ are greater than $\gamma^{(2)}$. Consideration of normal isotropic materials shows that $\gamma^{(2)}$ is roughly $1.5b^{(2)}$. Thus, the Rayleigh wave loci from equations (7:9a and b) can not exist in sector 1. Indeed, if $\gamma^{(2)}$ is at least $1.18b^{(2)}$ then $n_x^{(i)}b^{(i)}$ $(i = 1,3)$ can never be less than $b^{(2)} \equiv 1$. It is, however, possible that either or both of (7:9b and c) possess a solution which lies in sector 1.

Core layered shells present the opposite condition. Both $b^{(1)}$ and $b^{(3)}$ are smaller than $b^{(2)}$. They are, in fact, presumed to be less than 0.86 since $\gamma^{(1)}$ and $\gamma^{(3)}$, as well, are less than 1. In this case all four of the degenerate cases of

$D_4 = 0$ may possess sector 1 solutions if $b^{(4)}$ and $b^{(3)}$ are not too divergent.

The general three layered shell, with velocity ratios characterized by inequalities (5:43), embraces the range of values on the $\bar{\mu}$-$b^{(2)}$ plane excluded by the previously discussed shell types. The Rayleigh-like frequency equations may be satisfied by sector 1 values of m if the value of $b^{(2)}$ is less than 1 or not too much larger than 1. The significant difference in the behavior of the frequency equation is in the fact that the Stoneley-like wave equations may be satisfied, for a given $b^{(2)}$, by two ranges of $\bar{\mu}$. Figure 7 shows this. As a consequence of the near similarity of the materials of the shell it is clear that there is a much greater possibility of actually realizing the Stonsley wave loci as a part of the asymptotic behavior of the frequency equation.

2. The Frequency Equation in Sector 2 for $\zeta \to \infty$

Passage from sector 1 to sector 2 is a formal process, enacted by simply specifying that only rays of slope m lying in sector 2 will be admitted to consideration. The frequency equation must be modified to a certain extent. A choice must be made concerning the type shell under consideration. Equations (5:41), (5:42), and (5:43) set the bounds for sector 2 for all cases under consideration. The requirement to specify the bounds of sector 2 is of importance in order to decide which of the $p^{(i)}$ and $s^{(i)}$ (i = 1,2,3) change from imaginary

to real numbers.

For sandwich shells only Bessel functions with $\underline{p}^{(i)}$ and $\underline{s}^{(i)}$ in their arguments are effected by the transition to sector 2. For any $m > b^{(i)}$, the Bessel functions involving these two radial wave numbers are always regular Bessel functions.

For core layered shells sector 2 may be bounded by the turning point lines

$$\begin{aligned} &b^{(1)} < m < \eta^{(1)}, \\ &b^{(2)} < m < b^{(1)}, \\ &b^{(2)} < m < b^{(3)}, \\ &b^{(4)} < m < b^{(3)}. \end{aligned} \tag{7:24}$$

For each of these four cases only the radial wave numbers $\underline{p}^{(i)}$ and $\underline{s}^{(i)}$, for which the superscript (i) is the same as the superscript of the lower bound of the range, is effected. As long as m is greater than the lower bound, the effected radial wave numbers are always real.

Finally, for general three layered shells, the sector 2 bounds are

$$\begin{aligned} &b^{(1)} < m < \eta^{(1)}, \\ &b^{(2)} < m < b^{(1)}, \\ &b^{(2)} < m < b^{(3)}, \\ &b^{(4)} < m < \eta^{(3)}. \end{aligned} \tag{7:25}$$

The remarks following (7:24) hold also for (7:25).

A. Sandwich Shells - Sector 2

Only column elements of the frequency equation determinant which contain Bessel functions in $\underline{p}^{(i)}$ and $\underline{s}^{(i)}$ need be

modified from the formulation given in (7:5). The appropriate asymptotic expansions of the Bessel functions are given as

$$\lim_{\bar{r}\to\infty}\begin{Bmatrix} Z_n(\bar{r}\gamma^{(i)}) \\ W_n(\bar{r}\gamma^{(i)}) \end{Bmatrix} \to \left(\frac{2}{\pi\bar{r}\gamma^{(i)}}\right)^{1/2}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}\left(\bar{r}\gamma^{(i)} - \frac{n\pi}{4\bar{r}} - \frac{\Omega\pi}{2\bar{r}}\right), \tag{7:26}$$

where the arguments

$$\varphi^{(i)} = \bar{r}^{(i)} - \frac{n\pi}{4\bar{r}} - \frac{\Omega\pi}{2\bar{r}}, \tag{a}$$

$$\psi^{(i)} = \bar{r}^{(i)} - \frac{n\pi}{4\bar{r}} - \frac{\Omega\pi}{2\bar{r}}, \tag{b} \tag{7:27}$$

will be of use.

Making the simplifications used previously to obtain (7:5), the appropriate form of the C_{ij} for columns 7, 8, 11, and 12 (the only ones affected for sandwich shells in sector 2 as opposed to sector 1) are given as

$$C_{47} = \frac{d_4^{(i)}\bar{k}}{d_5^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\varepsilon^{(i)2} - d_4^{(i)}\frac{m^2}{\bar{r}^{(i)2}} - d_5^{(i)}\right]\varepsilon^{(i)}\bar{k}\sin\psi^{(i)}\bar{k},$$

$$C_{48} = \frac{-d_5^{(i)}\bar{k}}{d_5^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\varepsilon^{(i)2} - d_4^{(i)}\frac{m^2}{\bar{r}^{(i)2}} - d_5^{(i)}\right]\varepsilon^{(i)}\bar{k}\cos\psi^{(i)}\bar{k},$$

$$C_{4,11} = C_{4,12} = 0,$$

$$C_{57} = C_{58} = 0, \tag{7:28}$$

$$C_{5,11} = -\frac{1}{2}d_6^{(i)}\bar{r}^{(i)2}\bar{k}^2\cos\varphi^{(i)}\bar{k},$$

$$C_{5,12} = -\frac{1}{2}d_6^{(i)}\bar{r}^{(i)2}\bar{k}^2\sin\varphi^{(i)}\bar{k},$$

$$C_{61} = \frac{d_4^{(i)}\bar{k}^2}{d_5^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\varepsilon^{(i)2} + d_5^{(i)}\frac{m^2}{\bar{r}^{(i)2}} - d_5^{(i)}\right]\cos\psi^{(i)}\bar{k},$$

$$C_{68} = \frac{d_4^{(i)}\bar{k}^2}{d_5^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\varepsilon^{(i)2} + d_5^{(i)}\frac{m^2}{\bar{r}^{(i)2}} - d_5^{(i)}\right]\sin\psi^{(i)}\bar{k},$$

$$C_{6,11} = C_{6,12} = 0,$$

$$C_{77} = \xi^{(2)} E \sin\psi^{(2)} E,$$

$$C_{78} = -\xi^{(2)} E \cos\psi^{(2)} E,$$

$$C_{7,11} = C_{7,12} = 0,$$

$$C_{87} = C_{88} = 0,$$

$$C_{8,11} = -\xi^{(2)} E \sin\varphi^{(2)} E, \qquad (7:28)\ (\text{cont})$$

$$C_{8,12} = \xi^{(2)} E \cos\varphi^{(2)} E,$$

$$C_{91} = \frac{-E}{d_5^{(2)} + d_6^{(2)}} \left[d_1^{(2)} \xi^{(2)2} - d_4^{(2)} \frac{m^2}{\xi^{(2)2}} + d_5^{(2)} \right] \cos\psi^{(2)} E,$$

$$C_{92} = \frac{-E}{d_5^{(2)} + d_6^{(2)}} \left[d_1^{(2)} \xi^{(2)2} - d_4^{(2)} \frac{m^2}{\xi^{(2)2}} + d_5^{(2)} \right] \sin\psi^{(2)} E,$$

$$C_{9,11} = C_{9,12} = 0.$$

The remaining elements of the affected columns of the frequency equation are formed from (7:28) according to the instructions contained in (5:37). Equations (7:28) and (7:5) comprise the elements of the frequency equation in sector 2 for sandwich shells when ζ approaches infinity. The frequency equation is still degenerate relative to n, however, the trigonometric arguments of the four affected columns retain a dependence on n. Nonetheless, equation (6:31) retains its rôle as the description of the two determinants to which the frequency equation degenerates.

The torsional equation $D_3 = 0$ takes the form

$$D_3 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_4^{(1)}\xi^{(1)} & 0 & -d_\zeta^{(1)}\xi^{(1)}\cos\varphi^{(1)}_1 & -d_\varphi^{(1)}\xi^{(1)}\sin\varphi^{(1)}_1 \\ 0 & 0 & -1 & 0 & -\sin\varphi^{(1)}_1 & \cos\varphi^{(1)}_1 \\ 0 & 0 & 0 & 1 & -\sin\varphi^{(2)}_2 & \cos\varphi^{(2)}_2 \\ 0 & 0 & 0 & -d_\varphi^{(2)}\xi^{(2)} & -d_\zeta^{(2)}\xi^{(2)}\cos\varphi^{(2)}_2 & -d_\varphi^{(2)}\xi^{(2)}\sin\varphi^{(2)}_2 \end{vmatrix}$$

Expanded, $D_3 = 0$ yields the following requirement on m, ζ, and the elastic constants

$$\tan\left\{\pi\zeta\left(\frac{d_4^{(2)}}{d_4^{(1)}}\right)^{1/2}\left(\frac{m^2}{\xi^{(2)2}}-1\right)^{1/2}\right\} = \frac{\sqrt{d_4^{(1)}d_\zeta^{(1)}\left(\frac{m^2}{\xi^{(1)2}}-1\right)}\left[\sqrt{d_4^{(1)}d_\varphi^{(1)}\left(1-\frac{m^2}{\xi^{(1)2}}\right)}+\sqrt{d_4^{(2)}d_\varphi^{(2)}\left(1-\frac{m^2}{\xi^{(2)2}}\right)}\right]}{d_4^{(2)}d_\zeta^{(2)}\left(\frac{m^2}{\xi^{(2)2}}-1\right)-\sqrt{d_4^{(1)}d_\zeta^{(1)}d_\varphi^{(1)}d_\varphi^{(2)}\left(1-\frac{m^2}{\xi^{(1)2}}\right)\left(1-\frac{m^2}{\xi^{(2)2}}\right)}} \tag{7:29}$$

This frequency equation was first written by Stoneley [81] for Love waves in three layered isotropic media, and possesses roots in sector 2 of the frequency spectrum. It was first shown by Love [93] that in a homogeneous surface layer, over-lying a homogeneous half-space of another material, shear waves could exist. These "Love" waves are of the SH type, causing a displacement field perpendicular to both the propagation direction and the surface normal. Subsequent analysis by Stoneley and others has shown sufficient conditions wherein such SH waves can exist in internal homogeneous layers.

For any ray in sector 2, it is apparent at a glance that the right side of equation (7:29) is a constant. The argument of the tangent function is also constant except for ζ. Consequently, regardless of the value of the right side of (7:29), as ζ approaches infinity, the tangent will

repeatedly pass through the value given by the right side. Thus, on any given ray in sector 2, equation (7:29) possesses an infinity of zeros, each demarking the passage of a branch of the frequency equation across the ray.

Denoting the right side of (7:29) as Φ_p, that is, the principal value of the arctangent, the equation can be written as

$$\left(\frac{m^2}{b^{(2)2}}-1\right)^{1/2} = \frac{1}{\zeta\eta}\left(\frac{d_4^{(2)}}{d_3^{(2)}}\right)^{1/2} \tan^{-1}\Phi_p$$

When $\zeta \to \infty$

$$m \to b^{(2)} \Rightarrow \Omega \to b^{(2)}\zeta \tag{7:30}$$

For $m > b^{(2)}$ the branches of $D_3 = 0$ that enter sector 2 run to the asymptote $m = b^{(2)}$. Thus, all of the torsional modes propagate with the speed of the least shear wave velocity of the three materials of the cylinder when the wavelength is very short.

Turning now to $D_4 = 0$, the following equation results when only the dominant terms in each column are retained:

$$D_4 \to D_6^{(2)} D_6^{(3)} D_9 = 0, \tag{7:31}$$

where $D_6^{(l)}$ $(l = 1,3)$ are given in equations (7:9) and D_9 is given by (7:32) on page 166.

It is evident that the passage of $\underline{p}^{(2)}$ into the real domain has created a displacement field in the center layer of the sandwich cylinder which is not entirely a decay field, hence the two previously obtained Stoneley-like wave equations are now coupled. Because of the trigonometric terms in columns 7 and 8 (see (7:28)), $D_9 = 0$ possesses an infinity of

$$D_9 = \begin{vmatrix} C_{41} & C_{43} & C_{47} & C_{48} & 0 & C_{4,10} & 0 & 0 \\ C_{61} & C_{63} & C_{67} & C_{68} & 0 & C_{6,10} & 0 & 0 \\ C_{71} & C_{73} & C_{77} & C_{78} & 0 & C_{7,10} & 0 & 0 \\ C_{91} & C_{93} & C_{97} & C_{98} & 0 & C_{9,10} & 0 & 0 \\ 0 & 0 & C_{10,7} & C_{10,8} & C_{10,9} & 0 & C_{10,11} & C_{10,12} \\ 0 & 0 & C_{12,7} & C_{12,8} & C_{12,9} & 0 & C_{12,11} & C_{12,12} \\ 0 & 0 & C_{13,7} & C_{13,8} & C_{13,9} & 0 & C_{13,11} & C_{13,12} \\ 0 & 0 & C_{15,7} & C_{15,8} & C_{15,9} & 0 & C_{15,11} & C_{15,12} \end{vmatrix} \qquad (7:32)$$

zeros for any value of m in sector 2. By expanding (7:32), arranging terms as was done previously with $D_3 = 0$, and assuming the outer layers are identical, the following expression results for $D_9 = 0$:

$$\tan^{-1}\xi^{(2)} = \frac{2\,\alpha^{(2)} H_3 H_6}{\xi^{(2)2}\beta_2^2 - H_3^2}, \qquad (7:33)$$

where

$$H_3(m) = a^{(2)}\left[d_4^{(2)}(d_1^{(2)}\xi^{(2)2} - d_4^{(2)}m^2/b^{(2)2} - d_5^{(2)})H_1(m) + (d_5^{(2)} + d_4^{(2)})H_2(m)\right],$$

$$H_6(m) = a^{(2)2}\left[d_4^{(2)}(d_1^{(2)} + d_5^{(2)}(m^2/b^{(2)2} - 1)/\xi^{(2)2})H_1(m) - (d_5^{(2)} - d_4^{(2)})(m^2/b^{(2)2} - 1)\xi^{(2)2}H_2(m)\right],$$

$$H_1 = \begin{vmatrix} C_{61} & C_{63} & C_{6,10} \\ C_{71} & C_{73} & C_{7,10} \\ C_{91} & C_{93} & C_{9,10} \end{vmatrix}, \qquad H_2 = \begin{vmatrix} C_{61} & C_{65} & C_{6,10} \\ C_{71} & C_{75} & C_{7,10} \\ C_{91} & C_{95} & C_{9,10} \end{vmatrix},$$

$$H_3 = \begin{vmatrix} C_{61} & C_{63} & C_{6,10} \\ C_{81} & C_{83} & C_{8,10} \\ C_{91} & C_{93} & C_{9,10} \end{vmatrix}, \qquad H_4 = \begin{vmatrix} C_{61} & C_{65} & C_{6,10} \\ C_{81} & C_{83} & C_{8,10} \\ C_{71} & C_{73} & C_{7,10} \end{vmatrix},$$

and

$$\lim_{\xi^{(2)}\to 0} \frac{(m^2/b^{(2)2} - 1)}{\xi^{(2)2}} = \frac{d^{(2)}(d_5^{(1)} - d_4^{(1)}) - (d_5^{(1)} + d_4^{(1)})^2}{d_4^{(1)}(d_5^{(1)} - d_4^{(1)})}.$$

All branches of the frequency equation represented by $D_9 = 0$ which enter sector 2 run toward the limit $\xi^{(2)} = 0$ as

$\gamma \to \infty$. This yields the locus

$$\Omega = \nu^{(4)}\gamma \tag{7:34}$$

in the same fashion as (7:30). The branches of both D_3 and D_9 run to the same locus.

Reviewing the sandwich cylinder turning point line inequalities; inequalities (5:41), it is possible for $D_6^{(i)}$ $(i = 1,3)$ to represent nondispersed mode curves in sector 2. If $b^{(i)} \sim \gamma^{(i)}$ or $b^{(i)} \sim \nu^{(4)}$, then $m_n^{(i)} b^{(i)}$ $(i = 1,3)$ as given by inequality (7:12) may certainly be less than $\gamma^{(i)}$, and $\nu^{(4)}$ is the sector 2 upper boundary.

The very notion of a nondispersive branch of the frequency equation lying above the locus of an infinite family of branch curves is very disagreeable. It, in fact, at once confounds the physical observation that the Rayleigh waves derive from the lowest branches of the frequency equation, and the requirement that branches can not cross one another.

It will now be shown that the Rayleigh frequency equations in the product (7:31) may represent terraces in the high order branch curves of the frequency equation $D_4 = 0$. Terraces are described as zones in the branch curves which have the approximate slope of Rayleigh wave equation loci when $Z = m_n^{(i)} b^{(i)}$ $(i = 1,3)$ and thence decrease to the asymptotic limit given by (7:34). It will further be shown that the degenerate frequency equation $D_9 = 0$ is only valid when m is approximately equal to $b^{(i)}$ and $\gamma \gg 1$.

The elements in D_4, for $\gamma \gg 1$, are listed in equations

(7:5) and (7:28). It is to be noted that only the arguments of the exponential or trigonometric terms vary with ζ when D_4 is written for a particular ray with slope m in sector 2. All remaining factors are constant and bounded. Denoting terms which are constant by C, terms which are exponentially large and small by exp(x) and exp(-x), respectively, and trigonometric terms by sin and cos, D_4 may be written schematically as follows:

$$D_4 = \begin{vmatrix} D_6^{(1)} & \underline{e^{-2x}} & 0 & 0 & 0 & 0 \\ \underline{e^{-2x}} & \begin{matrix} C & C & \underline{e^{-x}} & 0 \\ C & C & \underline{e^{-x}} & C \end{matrix} & \begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ \underline{e^{-2x}} & \begin{matrix} C & C & \underline{e^{-x}} & C \\ C & C & \underline{e^{-x}} & C \end{matrix} & \begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & C & \underline{e^{-x}} \\ 0 & 0 & C & \underline{e^{-x}} \end{matrix} & \begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix} & \begin{matrix} C & C \\ C & C \end{matrix} & \underline{e^{-2x}} \\ 0 & \begin{matrix} 0 & 0 & 0 & \underline{e^{-x}} \\ 0 & 0 & C & \underline{e^{-x}} \end{matrix} & \begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix} & \begin{matrix} C & C \\ C & C \end{matrix} & \underline{e^{-2x}} \\ 0 & 0 & 0 & \underline{e^{-2x}} & D_6^{(3)} \end{vmatrix},$$

where the broken line encircles the elements which comprise D_9 and the "terms" outside this boundary are 2X2 determinants which, excepting $D_6^{(i)}$ (i = 1,3) indicate only the order of magnitude of each such determinant.

Certainly, when $|x| \to \infty$, only $D_6^{(1)}$, $D_6^{(3)}$, and D_9 remain. The underlined elements are a sort of "soft" coupling between the three diagonal determinants. If D_4, given above, is expanded for the purpose of grouping order of magnitude terms, and only terms no smaller than exp(-2x) are retained,

that is $|x|$ is assumed large enough to exclude all but the weakest coupling, D_4 becomes

$$D_4 = D_6^{(i)} D_4^{(j)} \left[D_5 + \sum c^6 e^{-\gamma} \sin \cos \right] + D_6^{(i)} \sum c^6 e^{-2x} \sin \cos + \\ + D_6^{(j)} \sum c^6 e^{-2x} \sin \cos \qquad (7:35)$$

Any value of γ and m which causes D_4 to vanish locates a point on a branch of D_4 in sector 2 of the frequency spectrum. Picking such a point when m is approximately $m_k^{(i)}$ and then varying m, so that with increasing γ, $D_4 = 0$ is always satisfied, a branch of the frequency equation can be followed through sector 2. For sandwich shells it is possible that $m_k^{(i)} b^{(i)}$ (i = 1,3) both lie in sector 2. This condition is assumed. It is asserted that as γ is increased, m decreases in order to obtain points on the same branch of the frequency equation. As m approaches the value $m_k^{(i)} b^{(i)}$ (assuming $m_k^{(i)} b^{(i)} \geq m_k^{(j)} b^{(j)}$ for definiteness), the first and second sums of (7:35) vanish individually due to the influence of $D_6^{(i)}$ and equation (7:35) reduces to

$$D_4 = D_4^{(j)} \sum c^6 e^{-2x} \sin \cos = 0$$

As m is further decreased, the sum $\sum c^6 \sin \cos e^{-2x}$ varies very slowly since the products of trigonometric terms tend to counteract each other and, moreover, e^{-2x} makes changes in magnitudes nearly imperceptible when $|x| \gg 1$. Consequently, as $m \to m_k^{(j)} b^{(j)}$ ($D_6^{(j)} \to 0$), equation (7:35) reduces to:

$$D_4 = D_6^{(i)} \sum c^6 e^{2x} \sin \cos = 0$$

Again the change of m as γ is increased is possibly very slow when m is nearly equal to $m_4^{(2)}b^{(2)}$. As γ is increased still further D_4 finally can be described accurately by D_9 alone, since, when $\gamma \to \infty$, the exponential factors vanish. Hence, the branch runs asymptotically to the slope $m = b^{(2)}$.

The coupling exhibited in (7:35) can never be ignored when m is not very close to $b^{(2)}$ since it is possible for either of the last two sums of this equation to play the dominant role in the satisfaction of the requirement that $D_4 = 0$. Thus, the weak coupling introduces the possibility of terracing, and also provides the conclusion that $D_9 = 0$ does not replace the requirement $D_4 = 0$ unless m is approximately $b^{(2)}$.

B. Core Layered Shells - Sector 2

Inequalities (7:24) delineate the bounds of sector 2. Observe that a duality in the four conditions exists. The ensuing discussion will illustrate quite clearly that only one set, for example,

$$b^{(2)} < m < \gamma b^{(2)} \quad , \quad b^{(1)} < m < b^{(2)}, \tag{7:36}$$

need be considered in detail. If (7:36) is employed, the sector 2 representation of the frequency equation will contain oscillatory factors in the elements of columns 13, 14, 17, and 18. If, on the other hand, the second two of (7:24) are used, the elements of columns 1, 2, 5, and 6 will be oscillatory. Because of this symmetry of the frequency equation, the results obtained by using (7:36) can be directly reinterpreted for the alternate case.

Equation (7:5) can be used for the elements of the frequency equation that do not change when moving from sector 1 to sector 2. Using the requirement of (7:36), the elements of columns 13, 14, 17, and 18 are obtained from (7:28) as follows:

$$C_{i,j}\ (i=10,11,\ldots,15;\ j=13,14,17,18) = -\,C_{i,j}\ (i=9,8,\ldots,4;\ j=7,8,11,12)$$ with superscript (2) replaced by (3), and $\bar{b}$ replaced by $\bar{c}$.

$$C_{i,j}\ (i=16,17,18;\ j=13,14,17,18) = C_{i,j}\ (i=15,14,13;\ j=13,14,17,18),$$ respectively, with $\bar{c}$ replaced by $\bar{d}$. (7:37)

From equations (7:5) and (7:37) it is apparent that the asymptotic form of the frequency equation again degenerates to the product D_3D_4. The determinant D_3 yields

$$D_3 = \left(\beta_b^{(2)}\bar{E}^{(2)} + \beta_c^{(3)}\bar{E}^{(3)}\right)\left(\beta_c^{(2)}\bar{E}^{(2)}\sin\varphi^{(2)}(\bar{d}\cdot\bar{z}) - \beta_c^{(2)}\bar{E}^{(2)}\cos\varphi^{(3)}(\bar{d}\cdot\bar{z})\right) = 0.$$

The first factor of this equation has no solution in sector 2, and the second factor reduces to

$$\tan\pi\gamma\,\frac{\beta_b^{(2)}}{\beta_c^{(2)}}\sqrt{\frac{\beta_b^{(2)}}{\beta_c^{(2)}}}\sqrt{\frac{m^2}{\beta^{(2)2}}-1} = \sqrt{\frac{\beta_c^{(2)}\beta_d^{(2)}}{\beta_c^{(3)}\beta_d^{(3)}}}\sqrt{\frac{1-m^2/\beta_c^{(2)2}}{m^2/\beta^{(3)2}-1}}\quad . \tag{7:38}$$

For isotropy, equation (7:38) reduces to the Love wave frequency equation. In the usual manner the branches represented by $D_3 = 0$ in sector 2 all asymptote to the sector boundary given as $m = \beta^{(2)}$ regardless of the value of n:

$$\Omega = \beta^{(2)}\gamma\ ,\quad \gamma \gg 1. \tag{7:39}$$

It is interesting to note that if the second qualification of (7:36) is employed, then in sector 3 the frequency equation $D_3 = 0$ may be written as

$$D_3 = \left(\beta_c^{(3)}\bar{E}^{(3)}\sin\varphi^{(3)}(\bar{k}\cdot\bar{z}) - \beta_c^{(3)}\bar{E}^{(3)}\cos\varphi^{(3)}(\bar{k}\cdot\bar{z})\right)\times$$
$$\times\left(\beta_b^{(3)}\bar{E}^{(3)}\sin\varphi^{(3)}(\bar{d}\cdot\bar{z}) - \beta_c^{(3)}\bar{E}^{(3)}\cos\varphi^{(3)}(\bar{d}\cdot\bar{z})\right) = 0. \tag{7:40}$$

Both the inner and outer layer are seen to have displacement fields which are similar to the Love wave field and the core displays a decay displacement field. This is reasonable since one may picture the wave front in the high velocity core as wheeling or turning to the outside layers and thus concentrating most of the core displacement at the interfaces.

$D_4 = 0$ takes the form

$$D_4 = D_6^{(1)} D_7^{(1,2)} D_{10}^{(1,2,3)} = 0, \tag{7:41}$$

where $D_6^{(1)}$ and $D_7^{(1,2)}$ are given by (7:9a and b), and

$$D_{10}^{(1,2,3)} = \begin{vmatrix} C_{10,7} & C_{10,9} & 0 & C_{10,14} & C_{10,15} & C_{10,16} \\ C_{12,7} & C_{12,9} & 0 & C_{12,14} & C_{12,15} & C_{12,16} \\ C_{13,7} & C_{13,9} & 0 & C_{13,14} & C_{13,15} & C_{13,16} \\ C_{15,7} & C_{15,9} & 0 & C_{15,14} & C_{15,15} & C_{15,16} \\ 0 & 0 & C_{16,8} & 0 & C_{16,15} & C_{16,16} \\ 0 & 0 & C_{18,8} & 0 & C_{18,15} & C_{18,16} \end{vmatrix}. \tag{7:42}$$

As concerns $D_6^{(1)}$ and $D_7^{(1,2)}$; only when the first of conditions (7:36) prevails is it possible that the effects of these determinants are not felt in sector 2. However, if the second of (7:36) holds, or if $b^{(1)}$ is sufficiently near $b^{(2)}$ in magnitude, $D_6^{(1)}$ and $D_7^{(1,2)}$ (if the materials of the inner layer and the core permit the existance of a Stoneley-like wave) may cause terracing of the branch curves in sector 2 at the values of m for which each of $D_6^{(1)}$ and $D_7^{(1,2)}$ vanish.

The frequency equation $D_{10}^{(1,2,3)} = 0$ reveals the locus of all branches when m is sufficiently close to $b^{(2)}$. This statement relies heavily on previous arguments and is supported here by the fact that if $D_{10}^{(1,2,3)}$ is expanded the

resulting terms have the form $\pm K(\cos\psi^{(0)}\bar{d}\cos\psi^{(0)}\bar{c} + \sin\psi^{(0)}\bar{d}\sin\psi^{(0)}\bar{c})$ or $\pm K(\sin\psi^{(0)}\bar{d}\cos\psi^{(0)}\bar{c} - \sin\psi^{(0)}\bar{c}\cos\psi^{(0)}\bar{d})$. Thus each term contains either $\sin\psi^{(0)}(\bar{d}-\bar{c})$ or $\cos\psi^{(0)}(\bar{d}-\bar{c})$. This consequence forces the conclusion that for any ray in sector 2 there are an infinite number of values of ζ for which $D_{10}^{(0)} = 0$. All modes must, then, asymptote to the sector lower bound given as

$$\Omega \to b^{(0)}\zeta \quad , \zeta \to \infty \, . \tag{7:43}$$

since they do not progress into sector 1.

For the alternate cases, posed by the last two of (7:24), the branches of $D_3 = 0$ and $D_4 = 0$ asymptote to the limit $m = b^{(0)}$.

C. General Shells - Sector 2

Recall that a great deal of symmetry was effected in writing the sector bounds for general three layered shells. The two facings are assumed materially identical and, as a result of this, the inequalities (5:43) result. For the pair of sector 2 bounds given as

$$b^{(0)} < m < b^{(0)},$$
$$b^{(0)} < m < b^{(0)},$$

the frequency equation in sector 2 is mathematically equivalent to the core layered cases discussed above with the exception that the two outer layers, being identical, cause the occurrence of equation (7:40) instead of (7:38) for $D_3 = 0$. Both factors of (7:40) represent the asymptotic locus $m = b^{(0)}$. The frequency equation $D_4 = 0$ degenerates into $D_{10}^{(0)} D_{10}^{(0)} = 0$, both of which represent the asymptote $m = b^{(0)}$.

As previously shown, the above forms of the frequency equation are valid only for m very near its asymptotic limit value and $\Psi \gg 1$.

For the cases

$$b^{(3)} < m < b^{(4)},$$

$$b^{(5)} < m < b^{(6)},$$

the sector 2 frequency equation is mathematically similar to the sector 2 frequency equation for sandwich shells. This being the case, $D_3 = 0$ yields the locus $m = b^{(3)}$ and $D_4 = 0$ yields the locus $m = b^{(4)}$ for the asymptotic limit of all branches.

3. The Frequency Equation for Higher Sectors

Having by this time become rather familiar with the general structure of the frequency equation, it can be shown now most easily that all branches of the frequency equation (5:34) pass into sector 2 on the Ω-Ψ plane and only a maximum of four ever pass below the sector 2 lower bound. The oscillatory nature of the regular Bessel functions may be generally regarded in the same manner as that of the sinusoidal functions. Consequently, in sector 7 of the frequency spectrum, where all Bessel functions are regular, any chosen ray must intersect an infinite number of branches excepting, possibly, those with zero cut-off frequency. In particular, the ray $m = \infty$ intersects all branches at their nonzero cut-off frequencies. When m attains the slope of the lower bound of sector 7, two columns (at least) of equation (5:34)

cease to be oscillatory. If, when this happens, (5:34) is expanded it is clear that there remain oscillatory factors in each term of the expansion. Further, ζ is the only variable in this expansion. Consequently, an infinite set of values for ζ may be found which depict the passage of the branches across the ray.

Progressing in a similar manner from sector 7 to 6, and so on to 3, at each sector lower bound, the expansion of equation (5:34) retains oscillatory factors in every term. There is always the possibility of selecting an infinite number of values for ζ which satisfy the requirement that the frequency equation vanish.

In sector 2 this behavior continues to hold until the slope of the ray, along which the examination is being carried out, attains the value of the sector 2 lower bound. When this occurs, all oscillatory components of (5:34) vanish, ζ no longer occurs as a free variable but appears implicitly in m. The detailed investigations in sector 2 have shown that all branches approach but do not cross the sector 2 lower bound. This, of course, is with exception to the few nondispersive branches which were shown explicitly to exist in sector 1.

With these facts in mind, it is clear that all branches progress into sector 2. One reasons as follows: when $m = \infty$, an infinity of branches may be found which cross the ray. As m decreases there is always an infinity of branches which are countable. An infinity of branches have been shown to

asymptote to the sector 2 lower bound slope, say m'. Suppose, however, some subgroup of branches of (5:34), emanating from nonzero cut-off frequencies, do not pass to the asymptote m'. Then, there is a slope m'' below which this subgroup does not pass. If the subgroup of branches is an interior group of the infinite family of branches of (5:43) then only a finite group of branches (those below the subgroup) pass to the limit slope m'. This, however, is not possible since it was shown that an infinite number of branches run to m'. Hence, there can be no such subgroup and all branches must pass to the asymptotic slope m'.

4. Summary of the Asymptotic Behavior of the Frequency Equation When $\xi \gg 1$

The most important result of the foregoing analysis is that, with the exception of some of the lower modes, all branches of the frequency equation approach the slope of the lowest turning point line for any choice of materials of the layers and for all $n \geq 0$. In other words, when the wavelength is very short, the traveling waves propagate with a phase velocity equivalent to the least shear wave velocity of the materials of the cylinder.

It has been shown that, for the case of isotropic layers, and for very small axial wavelength, the phase velocities of at most two modes can approach the Rayleigh wave speed for waves traveling in semi-infinite continua composed of the same material as the inner or outer layers. Moreover,

if the material constants comply with requirements similar to those set forth by Figure 7, Stoneley waves may exist. Then it is possible that the phase velocities of at most two additional modes can approach asymptotically the wave speeds for Stoneley waves traveling at the interface between semi-infinite bodies composed of the materials of layers one and two or three and two, when $\xi \to \infty$. Note Bene: pure Rayleigh or Stoneley waves can exist only in sector 1 of the $\Omega - \xi$ plane.

Three distinct cases may be considered.

a) The upper boundary of sector 1 is the turning point line with slope $\bar{b}^{(3)}$. For this case there is always a Rayleigh wave at the surface of the inner layer, traveling with the velocity $m_R^{(3)}\bar{b}^{(3)}$. If $m_R^{(3)}\bar{b}^{(3)}$ is greater than $\bar{b}^{(3)}$, no Rayleigh wave exists in the outer layer of the shell except as the successively higher modes terrace on the Rayleigh slope in passing to the asymptotic limit $\bar{b}^{(3)}$. If $m_R^{(3)}\bar{b}^{(3)}$ is less than $\bar{b}^{(3)}$ then there will be a Rayleigh wave at the surface of the outer layer for the case $\xi \to \infty$. The lowest branch of the frequency equation will approach asymptotically the line with least slope $m_R^{(i)}\bar{b}^{(i)}$ ($i = 1$ or 3) and the second branch will approach the remaining slope. If the material parameters are such that Stoneley waves can exist at the interface $\bar{r} = \bar{b}$, the phase velocity of the second or third mode will approach that of the Stoneley wave velocity depending on the ordering of this velocity relative to the two Rayleigh wave velocities. If the

materials of layers two and three will allow the presence of Stoneley waves, then the second, third, or fourth mode will approach the Stoneley wave velocity for the materials at the interface $\bar{r} = \bar{c}$. This, in turn, depends on the relative magnitude of the Stoneley velocity at this interface relative to the previously mentioned three velocities. If, on the other hand, $m_k^{(3)} b^{(3)}$ is greater $b^{(1)}$, there can still be a Stoneley wave in sector 1 for the $\bar{r} = \bar{c}$ interface of the shell since the Stoneley wave velocity is always bracketted by the least Rayleigh wave velocity and the least sheer velocity of either material involved. Since, it is possible for $m_k^{(2)} b^{(2)}$ to be less than $b^{(1)}$, it is possible for a Stoneley wave to exist at $\bar{r} = \bar{c}$.

b) The upper boundary of sector 1 is the turning point line with slope $b^{(3)}$. In this case the observations of paragraph (a), above, hold if the rôles of layers one and three are interchanged.

c) The upper boundary of sector 1 is the turning point line with slope $b^{(1)}$. When this is the case there will be no modes which approach the Rayleigh wave speeds if both $m_k^{(i)} b^{(i)}$ $(i = 1,3)$ are greater than $b^{(2)}$. The lowest two modes may, however, approach the Stoneley wave velocities at $\bar{r} = \bar{b}$ and $\bar{r} = \bar{c}$, providing the material constants of the layers allow this possibility (see Figure 7). If Stoneley waves exist at either interface, the velocities will necessarily be intermediate to $m_k^{(1)} b^{(1)}$ and $b^{(2)}$.

If $m_k^{(1)} b^{(1)}$ is less than $b^{(2)}$, a mode will approach the

Rayleigh wave velocity for a semi-infinite body composed of the material of the inner layer. The mode which approaches this velocity will be the first, second or third depending on the ordering of the two Stoneley wave velocities (if they exist) and the Rayleigh wave velocity. Similar observations hold if only $m_k^{(i)} b^{(3)}$ is less than $b^{(2)}$. If both $m_k^{(i)} b^{(i)}$ (i = 1,3) are less than $b^{(2)}$, then two modes will asymptote to the Rayleigh wave velocities of the inner and outer layer materials. In this case the lowest four modes will approach the four velocities noted (provided the Stoneley waves can exist) and their ordering will be dictated by the ordering of magnitudes of the four nondispersive velocities.

CHAPTER VIII

1. The Frequency Equation for Infinite, Three Layered Plates

If the mean radius of the cylinder is made infinitely large, the cylinder passes over into a three layered plate. If this limit is taken on the frequency equation it can be shown, also, to pass over to the frequency equation for the plate. To show that this is indeed the case, it is necessary to use the asymptotic large argument formulas for the Bessel functions, expand the determinant, and collect terms in factors which reduce to cos(thickness x wave number) and sin(thickness x wave number). With an 18X18 determinant the elegance of such a limiting process loses most of its luster and a more primitive method will be employed.

Using the transformation

$$dx = dr,\ dy = r\,d\theta,\ dz = dz, \qquad (8:1)$$

the requirement that $r \to \infty$ imposes that $d/dy \to 0$, eliminating all dependence on y. The coordinate system in the cylinder is replaced by that system shown in Figure 8. Although the dimension, e, is now zero, the notation, e, will be retained (formally) to preserve the similarity between the forthcoming and previously obtained equations.

The displacement field reduces to the conditions for

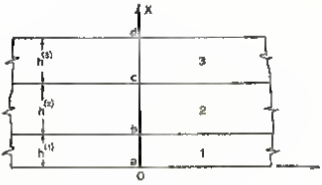

FIGURE 8. Plate Cross-Section

straight crested waves, given by:

$$u_x^{(i)} = \frac{\eta}{h^{(i)}}\left[\frac{\pi}{h^{(i)}}\bar{z}\,\frac{d\varphi_1^{(i)}}{d\bar{x}} + C_{44}^{(i)}\left(\varphi_5^{(i)} + d_4^{(i)}\right)\frac{d\varphi_5^{(i)}}{d\bar{x}}\right]\cos(\gamma\bar{y} - \Omega\bar{t}), \qquad \text{(a)}$$

$$u_y^{(i)} = 0, \qquad \text{(b)} \qquad (8:2)$$

$$u_z^{(i)} = \frac{\eta}{h^{(i)}}\left[\frac{\pi}{h^{(i)}}\bar{z}^2\left(d_1^{(i)}\epsilon_1^{(i)}\varphi_1^{(i)2} - d_2^{(i)}\frac{\bar{z}^2}{h^{(i)2}} + d_3^{(i)}\right)\frac{\varphi_1^{(i)}}{d\bar{x}^2} + d_4^{(i)}\right) +$$

$$+ C_{44}^{(i)}\bar{z}\left(d_1^{(i)}\epsilon_2^{(i)}\varphi_5^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{h^{(i)2}} + d_4^{(i)}\right)\varphi_5^{(i)}\Big]\sin(\gamma\bar{y} - \Omega\bar{t}), \qquad \text{(c)}$$

from (5:24), or

$$u_x^{(i)} = u_x^{(i)} = 0, \qquad \text{(a)} \qquad (8:3)$$

$$u_y^{(i)} = -\frac{\pi}{h^{(i)}}\frac{df^{(i)}}{d\bar{x}}\sin(\gamma\bar{y} - \Omega\bar{t}), \qquad \text{(b)}$$

from (3:57). This illustrates the fact that the displacement field is uncoupled in the effects from $\varphi^{(i)}(\bar{x})$ and $f^{(i)}(\bar{x})$. This uncoupling, which occurs for the cylinder only if $n = 0$, is seen here to be the only possible case (for straight crested waves), indicating that the nonaxisymmetric coupled modes of wave propagation in the cylinder have no counterpart in the plate theory. In the plate, the displacement

field (8:2) is a superposition of p and SV wave motion (dilatational and equivoluminal), whereas the field (8:3) represents SH wave motion (equivoluminal).

The stress components, when $\bar{r} \to \infty$, reduce to

$$\tau_{xx}^{(i)} = c_{44}^{(i)}\left(\frac{\eta}{k^{(i)}}\right)^2\left[-\frac{\eta}{k^{(i)}}\frac{d_4^{(i)}\xi^2}{d_5^{(i)}+d_4^{(i)}}\left(d_1^{(i)}\epsilon_1^{(i)}\varepsilon^{(i)2}+d_5^{(i)}\frac{\bar{\xi}^2}{k^{(i)2}}-d_5^{(i)}\right)\varphi_1^{(i)}-\right.$$

$$\left.-c_{44}^{(i)}d_4^{(i)}\xi^2\left(d^{(i)}\epsilon_3^{(i)}\beta^{(i)2}+d_5^{(i)}\frac{\bar{\xi}^2}{k^{(i)2}}-d_5^{(i)}\right)\varphi_2^{(i)}\right]\cos(\xi\bar{x}-\Omega\bar{t}),$$

$$\tau_{xy}^{(i)} = c_{44}^{(i)}\left(\frac{\eta}{k^{(i)}}\right)^2 d_6^{(i)}\epsilon_3^{(i)}\xi^2\,\varepsilon^{(i)2}\,\beta^{(i)}\sin(\xi\bar{x}-\Omega\bar{t}), \qquad (8:4)$$

$$\tau_{xz}^{(i)} = c_{44}^{(i)}\left(\frac{\eta}{k^{(i)}}\right)^2\left[\frac{\eta}{k^{(i)}}\frac{d_4^{(i)}\xi}{d_5^{(i)}+d_4^{(i)}}\left(d_1^{(i)}\epsilon_1^{(i)}\varepsilon^{(i)2}-d_4^{(i)}\frac{\bar{\xi}^2}{k^{(i)2}}-d_5^{(i)}\right)\frac{d\varphi_1^{(i)}}{d\bar{x}}+\right.$$

$$\left.+c_{44}^{(i)}d_4^{(i)}\xi\left(d_1^{(i)}\epsilon_1^{(i)}\beta^{(i)2}-d_4^{(i)}\frac{\bar{\xi}^2}{k^{(i)2}}-d_5^{(i)}\right)\frac{d\varphi_2^{(i)}}{d\bar{x}}\right]\sin(\xi\bar{x}-\Omega\bar{t}).$$

The differential equations (5:21) transform to

$$\left(\frac{d^2}{d\bar{x}^2}+\bar{\xi}^2\right)\left(\frac{d^2}{d\bar{x}^2}+\Omega^2\right)\varphi(\bar{x}) = 0, \qquad (a)$$

and (8:5)

$$\left(\frac{d^2}{d\bar{x}^2}+\xi^2\right)\bar{\psi} = 0, \qquad (b)$$

where $\bar{x} = x\eta/k^{(2)}$, as before.

The radial wave numbers retain their definitions listed in (5:12) and (5:19) with their appended equations.

The solutions for the potentials are, then, suitably specified as

$$\varphi_1^{(i)} = \bar{A}_1^{(i)}\mathcal{J}_0(\xi\varepsilon^{(i)}\bar{x}) + \bar{B}_1^{(i)}\mathcal{H}_0(\xi\varepsilon^{(i)}\bar{x}),$$

$$\varphi_2^{(i)} = \bar{A}_2^{(i)}\mathcal{J}_0(\xi\beta^{(i)}\bar{x}) + \bar{B}_2^{(i)}\mathcal{H}_0(\xi\beta^{(i)}\bar{x}), \qquad (8:6)$$

$$\bar{\psi}^{(i)} = \bar{A}_3^{(i)}\mathcal{J}_0(\xi\varepsilon^{(i)}\bar{x}) + \bar{B}_3^{(i)}\mathcal{H}_0(\xi\varepsilon^{(i)}\bar{x}),$$

where

$$\mathcal{Z}_0(\gamma^{(i)}\bar{z}) = \begin{cases} \cos\gamma^{(i)}\bar{z} & \text{if } \epsilon_j^{(i)} = 1, \\ \exp(\gamma^{(i)}\bar{z}) & \text{if } \epsilon_j^{(i)} = -1, \end{cases}$$

and

$$\mathcal{W}_0(\gamma^{(i)}\bar{z}) = \begin{cases} \sin\gamma^{(i)}\bar{z} & \text{if } \epsilon_j^{(i)} = 1, \\ \exp(-\gamma^{(i)}\bar{z}) & \text{if } \epsilon_j^{(\omega)} = -1. \end{cases} \qquad (j = 1,2,3) \qquad (8:7)$$

This choice of solutions conforms to the large argument asymptotic behavior of Z_0 and W_0. It will be convenient if appropriate differentiation formulas for (8:7) can be provided so that two separate developments (for $\epsilon_j^{(i)} > 0, \epsilon_j^{(i)} < 0$) of the boundary and interface conditions do not have to be made. These are found to be

$$\frac{d\mathcal{Z}_0}{d\bar{z}} = -\epsilon_j^{(i)}\gamma^{(i)}\mathcal{Z}_1(\gamma^{(i)}\bar{z}),$$

and

$$\frac{d\mathcal{W}_0}{d\bar{z}} = -\gamma^{(i)}\mathcal{W}_1(\gamma^{(i)}\bar{z}), \qquad (8:8)$$

with

$$\mathcal{Z}_1 = \begin{cases} \sin\gamma^{(i)}\bar{z} & \text{if } \epsilon_j^{(i)} = 1 \\ \exp\gamma^{(i)}\bar{z} & \text{if } \epsilon_j^{(i)} = -1, \end{cases}$$

and

$$\mathcal{W}_1 = \begin{cases} -\cos\gamma^{(i)}\bar{z} & \text{if } \epsilon_j^{(i)} = 1 \\ \exp(-\gamma^{(i)}\bar{z}) & \text{if } \epsilon_j^{(i)} = -1, \end{cases} \qquad (j = 1,2,3) \qquad (8:9)$$

Note the similarity of (8:8) with the Bessel function differentiation formulas (5:32).

Using equations (8:7) and (8:8) with (8:6) and (5:31) in (8:2), (8:3), and (8:4) yields the basic set of equations from which the frequency equation is formed. As pointed out earlier

$$\lim_{F\to\infty}(i)\ \det|C_{ij}| = D_3\,D_4\,, \tag{8:10}$$

where the elements of D_3 and D_4 are listed formally in equation (6:15). For the plate, these elements are given (in the format of (5:37) so that the instructions therein may be applied here) as

$$C_{11} = \frac{-d_4^{(i)}\gamma^3}{d_3^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} + d_3^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\mathcal{Y}_0(\gamma\xi^{(i)}\bar{a}),$$

$$C_{12} = \frac{-d_3^{(i)}\gamma^3}{d_3^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} + d_3^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\mathcal{Y}_0(\gamma\xi^{(i)}\bar{a}),$$

$$C_{13} = -d_4^{(i)}\gamma^2\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_3^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\mathcal{Y}_0(\gamma\beta^{(i)}\bar{a}),$$

$$C_{14} = -d_4^{(i)}\gamma^2\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} + d_3^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\mathcal{Y}_0(\gamma\beta^{(i)}\bar{a}),$$

$$C_{15} = C_{16} = 0,$$

$$C_{21} = C_{22} = C_{23} = C_{24} = 0,$$

$$C_{25} = d_6^{(i)}\epsilon_3^{(i)}\gamma^2\xi^{(i)2}\mathcal{Y}_0(\gamma\xi^{(i)}\bar{a}),$$

$$C_{26} = d_6^{(i)}\epsilon_3^{(i)}\gamma^2\xi^{(i)2}\mathcal{Y}_0(\gamma\xi^{(i)}\bar{a}), \tag{8:11}$$

$$C_{31} = \frac{-d_4^{(i)}\epsilon_1^{(i)}\gamma^2}{d_3^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\gamma\xi^{(i)}\mathcal{Y}_1(\gamma\xi^{(i)}\bar{a}),$$

$$C_{32} = \frac{-d_3^{(i)}\gamma^2}{d_3^{(i)}+d_4^{(i)}}\left[d_1^{(i)}\epsilon_1^{(i)}\xi^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\gamma\xi^{(i)}\mathcal{Y}_1(\gamma\xi^{(i)}\bar{a}),$$

$$C_{33} = -d_4^{(i)}\epsilon_2^{(i)}\gamma\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\gamma\beta^{(i)}\mathcal{Y}_1(\gamma\beta^{(i)}\bar{a}),$$

$$C_{34} = -d_4^{(i)}\gamma\left[d_1^{(i)}\epsilon_2^{(i)}\beta^{(i)2} - d_4^{(i)}\frac{\bar{z}^2}{\xi^{(i)2}} - d_5^{(i)}\right]\gamma\beta^{(i)}\mathcal{Y}_1(\gamma\beta^{(i)}\bar{a}),$$

$$C_{35} = C_{36} = 0,$$

$$C_{41} = -\epsilon_1^{(i)}\gamma^2\xi^{(i)}\mathcal{Y}_1(\gamma\xi^{(i)}\bar{a}),$$

$$C_{12} = -\gamma^2 \alpha_1^{(1)} \mathcal{H}_1(\gamma \alpha_1^{(1)} b),$$

$$C_{13} = -(d_3^{(1)} + d_4^{(1)})\, \epsilon_2^{(1)} \gamma \beta^{(1)} \mathcal{J}_1(\gamma \beta^{(1)} b),$$

$$C_{14} = -(d_3^{(1)} + d_4^{(1)})\, \gamma \beta^{(1)} \mathcal{H}_1(\gamma \beta^{(1)} b),$$

$$C_{15} = C_{16} = 0,$$

$$C_{21} = C_{22} = C_{23} = C_{24} = 0,$$

$$C_{25} = \epsilon_3^{(2)} \gamma \xi^{(2)} \mathcal{J}_1(\gamma \xi^{(2)} b), \qquad \text{(8:11) (cont)}$$

$$C_{26} = \gamma \xi^{(2)} \mathcal{H}_1(\gamma \xi^{(2)} b),$$

$$C_{31} = \frac{\gamma^2}{d_3^{(1)} + d_4^{(1)}} \left[d_1^{(1)} \epsilon_1^{(1)} \alpha_1^{(1)2} - d_4^{(1)} \frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)} \right] \mathcal{J}_0(\gamma \alpha_1^{(1)} b),$$

$$C_{32} = \frac{\gamma^2}{d_3^{(1)} + d_4^{(1)}} \left[d_1^{(1)} \epsilon_1^{(1)} \alpha_1^{(1)2} - d_4^{(1)} \frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)} \right] \mathcal{H}_0(\gamma \alpha_1^{(1)} b),$$

$$C_{33} = \gamma \left[d_1^{(1)} \epsilon_2^{(1)} \beta^{(1)2} - d_4^{(1)} \frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)} \right] \mathcal{J}_0(\gamma \beta^{(1)} b),$$

$$C_{34} = \gamma \left[d_1^{(1)} \epsilon_2^{(1)} \beta^{(1)2} - d_4^{(1)} \frac{\Sigma^2}{k^{(1)2}} + d_4^{(1)} \right] \mathcal{H}_0(\gamma \beta^{(1)} b),$$

$$C_{35} = C_{36} = 0.$$

The remaining C_{ij} are formed from the above by use of the instructions contained in (5:37).

For the case of identical materials of the three layers of the plate, and with H as the total thickness, the equation $D_3 = 0$ decomposes into the two frequency equations for symmetric and antisymmetric face shear waves. These equations are, respectively,

$$\sin(\gamma \xi H) = 0, \qquad \text{(a) (8:12)}$$

$$\cos(\zeta \mathfrak{S} H) = 0 \qquad \text{(b) (8:12)}$$

The equation $D_4 = 0$, under similar conditions, degenerates into the frequency equations for symmetric and antisymmetric Rayleigh-Lâme waves given by

$$\frac{\tan(\tau \mathfrak{S} H)}{\tan(\zeta \mathfrak{S} H)} = -\left[\frac{4 \mp \beta \zeta^2}{(\zeta^2 - \alpha^2)^2}\right]^{\pm 1}, \qquad (8:13)$$

where the +1 is for symmetric waves and -1 is for the antisymmetric case. These wave equations have been discussed fully by Mindlin [72] and Meeker and Meitzler [94].

Equations (8:12) are quite similar to the elements which compose (6:15), hence the behavior of (8:11) and (6:15) should be comparable. Furthermore, when $\zeta \to \infty$, equations (6:15) pass over into the identical form as (8:12) which bears out the conclusion that when the radial wave number is very large the wave phenomena in the cylinder approach the plate as a limiting case. There is, consequently, no need to discuss the asymptotic behavior of (8:11).

CHAPTER IX

1. Method of the Numerical Analysis

A computer program, written for execution on an IBM 7044/7094 DOS System, and using the modus described in the opening paragraphs of Chapter VII, was used for obtaining a numerical evaluation of the frequency equation (5:34) and its degenerate forms (6:7), (6:14), and/or (6:21).

The program, consisting of a calling program and 30 subroutines, was devised in a manner such that any aspect of the problem could be successfully treated regardless of the degree of anisotropy dictated by the material constants. Previously written and tested subroutines for computing Bessel functions of integer order and real, imaginary, or complex arguments; and subroutines for determinant evaluation and matrix equation solution were incorporated from sources other than the author's pen.

Use of the "ray" analysis furnishes a time saving factor in the problem execution since, once the angle of the ray is specified, and the sector of the Ω-γ plane in which it lies identified, there is no requirement to reascertain this information on each successive iteration of Ω or γ until a new ray is specified. Furthermore, those quantities depending only on the ratio Ω/γ (for instance, $s^{(\cdot)}$, $\beta^{(\cdot)}$,

$\xi^{(0)}$ need only be determined at the outset of computations for each new ray.

The program, in normal sequence, executes the cut-off ($\tau = 0$) computations - in which roots of the determinants $\tilde{D}_1$, $\tilde{D}_3$, and D_5 for $n = 0$, or D_3 and D_4 for $n > 0$ are found and displacement distributions determined. Subsequently, starting at a preset minimum value of η or ζ, the frequency equations D_3 and D_4 for $n = 0$ or $\det|C_{ij}|$ for $n > 0$ for finite wavelength ($\tau \neq 0$) are examined along rays emanating from the origin. The starting angle of the ray with steepest slope may be any prechosen value. For the curves presented in Section 2 of this chapter a maximum angle of 88 degrees to the ζ-axis was chosen. Each successive ray is declined at 2 degree increments until the angle 60 degrees is obtained. Below this angle successive radials one degree apart are examined until a radial of sufficiently shallow angle is examined such that the frequency equation has no zeros. The density of radials examined is controlled by input data.

Along each radial the zeros of the determinants listed above are detected by sign change information of the determinants. When a sign change of the determinant is noted for two consecutive values of the variables, the increment is halved. A test is executed to determine which half interval the root lies in and this interval is subsequently tenthed until a sign change in the determinant value reoccurs. Coordinates of the zeros thus bracketted are then

deemed to be midway between these two smallest increments in Ω or ζ over which the sign change occurred. Since the maximum size of the normal increment is controlled by input data, the veracity of the roots so located can be controlled to any degree of accuracy sought. For the figures presented herein, the accuracy of Ω and ζ is controlled to within $\pm$ 0.0025 - a quantity deemed sufficiently small for plotting accuracy.

The following scheme was employed to regulate the increment length along a ray - a distance which is always greater than the increment set off along the coordinate axes. The variable, either Ω or ζ, which is chosen as the independent variable along a given ray is determined by the inclination of the ray to the ζ-axis. For rays of inclination greater than 60 degrees, Ω is the independent variable with a normal increment of $\Delta\Omega$. For rays of inclination 45 degrees to 60 degrees, Ω is again the independent variable and $\Delta\Omega/2$ the normal increment. Similarly, for rays less than 45 degrees, ζ is the independent variable and the increment is $\Delta\zeta = \Delta\Omega$ for rays of angle less than 30 degrees or $\Delta\zeta/2$ for rays between 30 degrees and 45 degrees.

The computation along a given ray is terminated when either the prescribed number of roots have been found, or a preset ray length has been obtained.

Special means have been written into the program so that localized high density variation of the variables

Ω and ζ and/or the ray slope may be obtained. This facility is required whenever two branches are very close together or when undispersed modes (which occur when $\zeta \gg 1$) are located.

The displacements, computed only for the cut-off cases, have been found in the following manner. For $n = 0$ the $A_i^{(2)}$ ($i = 1,2$, or 3) of the middle layer, appearing in $\bar{D}_1$, $\bar{D}_3$, or D_5, has been arbitrarily set to 1.0. For the case $n > 0$ the integration constants $A_2^{(2)}$ and $A_1^{(2)}$ were chosen as 1.0 for the plane-strain and the longitudinal shear displacement distribution, respectively. The distributions of displacement, once found, are normalized relative to the largest amplitude of the particular distribution so that in each instance comparisons can be more easily made from one case to another. To further facilitate comparison of similar modes of vibration, the displacements for each different case of geometry and/or materials is displayed with the corresponding mode for a homogeneous cylindrical shell composed of the same material as the core layer and total thickness of the composite shell.

The program was checked by comparison of output with Gazis' [21] results for the case $n = 1$, and comparison with Mirsky's [40] results for the case $n = 0$. Although both test cases were for homogeneous shells, it is easily recognized that algebraic or sign errors, if present in the three layer shell equations, would eliminate consistant and accurate results. To eliminate the remote possibility of

self rectifying errors, a third test case for two layered composite shells was also employed. This case was furnished by Dr. A. E. Armenàkas by private arrangement.

The program described above was written exclusively in double precision and left less than 40 unused locations in the 7094 core memory. For this reason it was necessary to write a second program for the express purpose of determining the displacement and stress distributions in the composite cylinder for the case of traveling waves ($\zeta \neq 0$). This was accomplished utilizing many of the subroutines previously employed - modified suitably to treat only the non-homogeneous set of equations which evolve for computation of the integration constants based on the preknowledge of the roots of the frequency equation. Due to computer time limitations, this program has been debugged, but not run for output in the dissertation.

The data governing the cases presented in Section 2 of this chapter are given below. The first data case is for Poisson's ratio $\nu = 0.3$ and all layers of the same material. This case reflects the conditions for a homogeneous shell and is included for the purpose of comparison with the three layer cases. In both three layer cases the inner and outer shells are treated as of the same thickness and material composition, and all three layers are isotropic with $\nu^{(i)} = 0.3$ $(i = 1,2,3)$. Moreover, results have been obtained only for the axisymmetric case $(n = 0)$.

homogeneous: $d_1^{(i)} = 3.5, \quad d_3^{(i)} = 3.5, \quad d_5^{(i)} = 1.5,$
$d_2^{(i)} = 1.5, \quad d_4^{(i)} = 1.0, \quad \zeta^{(i)} = 1.0,$ $(i = 1,2,3)$

Case I		Case II	
$d_1^{(i)} = 3.5$	$d_1^{(o)} = 3.5$	$d_1^{(i)} = 7.0$	$d_1^{(o)} = 3.5$
$d_2^{(i)} = 1.5$	$d_2^{(o)} = 1.5$	$d_2^{(i)} = 3.0$	$d_2^{(o)} = 1.5$
$d_3^{(i)} = 3.5$	$d_3^{(o)} = 3.5$	$d_3^{(i)} = 7.0$	$d_3^{(o)} = 3.5$
$d_4^{(i)} = 1.0$	$d_4^{(o)} = 1.0$	$d_4^{(i)} = 2.0$	$d_4^{(o)} = 1.0$
$d_5^{(i)} = 1.5$	$d_5^{(o)} = 1.5$	$d_5^{(i)} = 3.0$	$d_5^{(o)} = 1.5$
$\zeta^{(i)} = 0.5$	$\zeta^{(o)} = 1.0$	$\zeta^{(i)} = 1.0$	$\zeta^{(o)} = 1.0$

For each set of material constants noted above the following two sets of geometric constants were employed: $H/R = 1.0$ and $H/R = 0.1$, where

$$H = h^{(i)} + h^{(c)} + h^{(o)}$$

defines the total thickness.

The first geometry corresponds to the case of a thick shell and the second to that of a thin shell. The thin shell case has a total thickness to radius ratio (that is, $H/R = 0.1$) which is within the range of validity of the thin shell bending theories [95,96]. Furthermore, it is expected that this thin shell case may fall within the range of validity of the sandwich theory developed by Yu [41,76,77].

Each combination of material constants versus H/R ratio was executed for the single choice of thin outer layers, as depicted by the ratio $h^{(i)}/h^{(o)} = 0.1$. Consequently, a total of six frequency spectra were computed, each showing the first ten branches of the axisymmetric frequency equation.

All of the frequency spectra have been plotted to a value of $\zeta = 5.0$. This value of ζ was chosen since it represents a value of the half wavelength L which is considerably smaller than any dimension of the shell. For instance, in terms of the outer diameter to half wavelength ratio, that is, d/L, the abscissa value of $\zeta = 5.0$ corresponds to $d/L = 62.9$ and $d/L = 9.0$, respectively, for the two geometries $H/R = 0.1$ and 1.0, when $h^{(i)}/R^{(i)} = 0.1$. The maximum value, $\zeta = 5.0$, is sufficiently large so that the outset of the asymptotic effects discussed in Chapter VII can be observed. This is reasonable since when $\zeta \sim 5$ it is evident that the shell thicknesses are beginning to assume very large proportions relative to the half wavelength of the axially traveling waves.

2. Discussion of the Frequency Spectra

Since the frequency spectra are plotted over such large ranges of values of the variables Ω and ζ, it has been found advantageous to plot the spectra in two separate figures (for each case). The first of any such set of figures pertains to the long wavelength region of the variables, and the second to the mid-wavelength, or almost asymptotic range of variables. The inserts in Figures 15, 17, and 19 are for the lowest branches of the frequency equation near the origin, and have been so included for the sake of clarity. Small portions of the ninth or higher branches have been excluded for the range of values of ζ near 2.5 due to the method of

presentation of the curves, but the omissions obscure no unusual behavior of the frequency equation. Due to the manner in which the computer program counted solutions of the frequency equation, fragments of the eleventh and twelfth branches were obtained. These fragments were included to complete the orientation of the branches in the cut-off region.

Cases I and II both fall in the class of general shells described by inequalities (5:43). The ordering of the turning point lines is such that

$$\eta^{(4)} > \eta^{(3)} > \eta^{(2)} > \eta^{(1)},$$

and

$$\eta^{(5)} = \eta^{(6)}, \quad \eta^{(5)} = \zeta^{(5)}.$$

Both case I and II yield the same values for the $\eta^{(5)}$ and $\eta^{(i)}(i = 1,2,3)$ and the spectra, therefore, present a means of comparing the separate effects of different stiffnesses as well as densities when these types of variation each lead to an analytically identical form for the frequency equation.

For ease of identification of the branches of the frequency equation the following notation has been used to identify the branches. Since only the axisymmetric condition is under study all branches originate with cut-off frequencies determined by solutions of $\tilde{D}_1$, $\tilde{D}_3$, and D_5. When $\gamma \neq 0$ the branches are determined by D_3 and D_4. Consequently, all branches emanating from the solutions of $\tilde{D}_3(\gamma = 0)$ and continued by solutions of $D_3(\gamma \neq 0)$ will be called torsional

branches and demarked $T_1, T_2, \ldots$, according to the order of the cut-off frequency of the torsional branch relative to other, lower torsional branches. In a similar manner the branches emanating from solutions of $\tilde{D}_1$ and D_5, and perpetuated by D_4 will be called nontorsional branches and demarked NT_{ij} where i indicates the order of the branch relative to nontorsional branches with smaller cut-off frequencies, and j is 1 if the branch is a longitudinal-shear mode at its cut-off frequency or 2 if radial plane-strain.

A study of the six spectra reveals the following generalities:

a) For $\zeta < 1$ the branches of the frequency equation are comparable for all cases of the same geometry, regardless of the material constants of the layers, and;

b) For $\zeta > 1$ the branches of the frequency equation are comparable, regardless of the geometry, for each of the three cases of material constants.

These observations indicate that the shell geometry plays a strong rôle in the determination of branches in the long wavelength region and this influence decreases as the wavelength decreases. Furthermore, the rôle of the material constants appears more pronounced in the short wavelength region.

Turning now to more specific observations, the long wavelength region will be considered initially. It is clear from the identification of the branches of each of the six spectra what the nature of the modes of vibration are at

the cut-off frequencies of each branch. The displacement distributions of the first six modes of vibration have been illustrated in Figures 21, 22, and 23 and will be discussed in Section 3.

Comparison of the three layer spectra with the homogeneous spectra indicates that the first longitudinal shear mode initiates with a phase velocity similar to the phase velocity, $\sqrt{E/\rho}$, for isotropic homogeneous shells. It is further noted that this branch is the only one which does not initiate in the highest sector of the Ω-ζ plane. Since the outer materials are identical there are only five sectors on the Ω-ζ plane and the NT_{11} branch always initiates in sector 3. This requires that the radial wave numbers $\xi^{(i)}$ $(i=1,2,3)$ be imaginary. Consequently, the dilatational potentials $\varphi_2^{(i)}$ $(i=1,2,3)$ contribute exponentially decaying components to both the wave equation and the displacement field.

The radial extensional branch NT_{42} has a pronounced minimum in each of the six spectra. Superficially, this trough indicates a negative phase velocity (see equation (5:39)). However, it is well known (see, for instance, [72]) that the above conclusion does not apply. Since the axial wave number ζ appears in the frequency equation only in squared form, it is immediately obvious that spectra similar to those exhibited in Figures 9 through 20 could be plotted for $-\zeta$. Furthermore, it has been shown [19] that at each local minimum of a branch (for real ξ) of the

frequency equation, another branch emanates which satisfies the frequency equation for real Ω and complex ζ. These new branches either decrease through complex values of ζ to another local minimum of a branch which is defined for real ζ only, or decrease to the complex ζ plane, for which $\Omega = 0$. In this manner it is possible, through branches of the frequency equation satisfied by complex, negative, or positive values of ζ, to begin with $\Omega = 0$ and define paths along the various branches, which, for continuous monotonically increasing Ω, always have zero or positive group velocity. Thus, that portion of NT_{42} (and any other branch, as well) with negative slope may be looked on as indicating that the mirror reflection, about the Ω axis, is the appropriate segment to be chosen in a complete description of the mode curves. Furthermore, it necessarily follows that the NT_{11} branch is the only branch of the frequency equation which initiates with the value $\zeta = 0$ when $\Omega = 0$. All other branches originate (at $\Omega = 0$) with some complex value of ζ.

The cut-off frequencies of all modes are raised moderately with increased density of the inner layer, and only slightly effected by increasing the stiffness of the outer layers. This is seen from comparison of Figures 11 and 13 with 9, or 17 and 19 with 15. As previously remarked, the geometry of the shell more extensively effects the cut-off frequencies and it is observed that as the H/R ratio decreases for a given set of material constants, both the T_i

and NT_{12} modes initiate at notably smaller cut-off frequencies, when compared with H/R = 1.0.

Examining next the region, of all spectra, for $\zeta > 1$, several interesting facts are revealed. Consider first the two sets of figures for the homogeneous cylinders, that is Figures 9, 10, 15, and 16. Two important results are immediately observable. The lowest nontorsional modes, NT_{11} and NT_{22} both run to the Rayleigh locus $\Omega = 0.927\zeta$. This locus is characterized by the ray inclined at an angle of 42.83 degrees to the ζ axis and is indicated in the figures by the broken line. Notice that NT_{11} approaches the Rayleigh slope from below and NT_{22} from above. The analysis conducted in Chapter VII, when suitably altered for homogeneous cylinders, indicates this behavior of both NT_{11} and NT_{22}. The lowest nontorsional mode with nonzero cut-off frequency (NT_{22}) approaches the Rayleigh locus as $\zeta \to \infty$. From the fact that NT_{11} never lies in the highest sector (sector 3 for homogeneous cylinders), a similar analysis easily yields the same conclusion for this branch of the frequency equation.

The second fact of significance with regard to the homogeneous cylinder spectra is the twinning of the T_i and $NT_{i+1,j}$ branches. It is noticed that decreasing H/R ratio brings these curves into contact when H/R = 0.1. Apparently, for H/R < 0.1 the T_i branches will drop below the $NT_{i+1,j}$ branches. It has been observed that the contact point between the branches for H/R = 0.1, and the locales of closest

approach for H/R = 1.0, fall along the ray $\Omega = \sqrt{2}\zeta$. This is the required relation of Ω to ζ for the existence of Lamè [51] modes in plates. However, Armenàkas [25] has observed that the Lamè modes can not exist in composite shells.

The spectra for three layered shells (Figures 11 through 14, and 17 through 20) show that no mode decreases below the lowest turning point line $\Omega = \zeta$ (recall that $b^{(1)} = 1$). This must be the case since, as was shown in Chapter VII, Rayleigh waves can exist only if $b^{(i)}\ (i = 1,3) < b^{(2)}$. Furthermore, the choice of material constants will not allow the existence of Stoneley interface waves.

Of paramount concern is the effect on the branches as the stiffnesses of the outer layer, or density of the core is increased. Increasing the core layer density retards the rate of approach of all branches to the limit $\Omega = \zeta$. Notice, as well, that all hint of twinning of the branches is removed. The extent of the approach of the upper branches to the asymptotic limit is not greatly effected by the value of H/R.

When comparing the spectra of Figures 13, 14, or 19, 20 with the corresponding homogeneous spectra, it is at once noticed that not only is the twinning tendency obviated, but furthermore, the T_i branches intersect, from below, the $NT_{i+2,j}$ branches. This is a pronounced difference in the small wavelength behavior of the mode curves. Comparison of the slopes of the two families of branches indicates

that increasing the outer layer stiffnesses results in reducing the rate of approach of the T_i curves to the limit $\Omega = \zeta$ and has little effect on the rate of approach of the NT_{ij} curves - when compared with the homogeneous spectra. Comparison of Figure 12 and 14 or 18 and 20 show that the NT_{ij} branches approach their asymptotic limit $\Omega = \zeta$ more rapidly in the shells with stiffer outer layers, and the T_i branches more slowly.

In summary it is pointed out that the cut-off frequencies are strongly effected by the ratio H/R - approaching the appropriate plate frequencies as $H/R \to 0$, and the solid cylinder frequencies (Pochhammer frequencies for the homogeneous case) as $H/R \to 2$. Furthermore, as the stiffness of the outer layers or density of the core, is increased, the rate of approach of the branches to the limit $\Omega = \zeta$ is retarded, relative to the homogeneous case; it being slightly more pronounced for small H/R than for large.

3. Discussion of the Vibration Displacement Distributions

The identification of the branches of the frequency spectra in Figure 9 through 20 indicate immediately the nature of the displacements when $\zeta = 0$. The branch curve NT_{11} does not represent a vibratory mode when ζ is zero since the frequency is also zero. The branch T_0, noted on the two homogeneous spectra, is for the classical first torsional mode given as $u_\theta = (Ar)\cos\Omega\tau$. This mode is not obtained from the solution given in

Chapter V for composite cylinders.

Since NT_{11} (for $\zeta = 0$) does not represent a mode of vibration, NT_{22} is the only mode, for composite shells, which does not possess a nodal cylinder. This is a so called ring mode [22] and has been termed the breathing mode. All longitudinal-shear and circumferential-shear modes possess one or more nodal cylinders and have been characterized as thickness modes [22]. Moreover, the second and higher radial plane-strain modes of vibration are also thickness modes.

The first two modes of each of the three types mentioned above are displayed in Figures 21, 22, and 23. The modes have been plotted in the manner previously discussed in Section 1 of this chapter. Only the two cases, for each geometry, governed by homogeneity and the case I material constants have been plotted, for reasons of clarity. The mode shapes for the case II material constants are generally bounded by the pairs of curves given in each figure.

In comparing the curves in each figure, of $H/R = 0.1$ and $H/R = 1.0$, it is immediately apparent that the mode shape curves show almost no attenuation in amplitude with respect to thickness for the ratio $H/R = 0.1$. This is directly the result of the proximity to plate geometry that this small value of H/R represents. Furthermore, an additional consequency of this approach to plate conditions is evinced by the symmetry of the distributions relative to the center plane of the shell for small H/R.

Figure 22, for H/R = 1.0, contains the only plot of e case II date displacement distribution. The case 1 distribution for this mode nearly coincides with the homogeneous case. Consequantly, compared to tha homogeneous case, the relative stiffnesses of ths layers influence the ring mode displacement distribution more greatly than the relative densities.

Study of the displacement distributions for all three cases of material constents, and for each case of geometry reveals that the effects of greater outer layer stiffnesses is similar to the effects of greater core density with regard to the thickness mode shapes near the surfaces of the cylinder. The outer layers of the composite shells have distributions with less curveture than for the outer extremities of the homogeneous shells. This, in the case of stiff outer layers, follows without explanation. For tha case of shells with more dense cores, tha result is due to tha greater inertia of the core layer which, on the basis that all three layers have ths same stiffness, implies the observed effect.

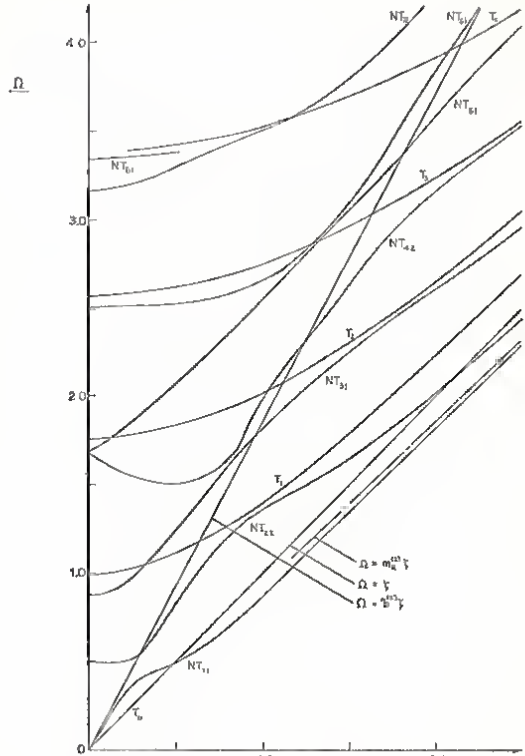

FIGURE 9. Frequency Spectrum: Homogen., H/R = 1.0

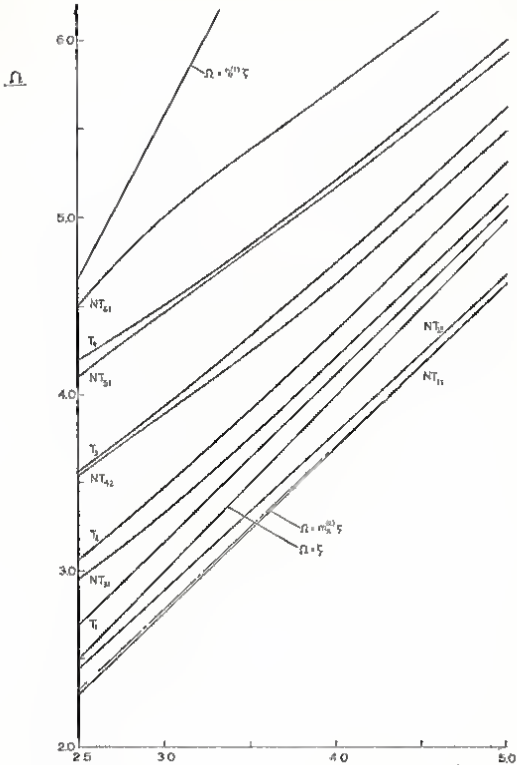

FIGURE 10. Frequency Spectrum: Homogen. (cont), H/R = 1.0

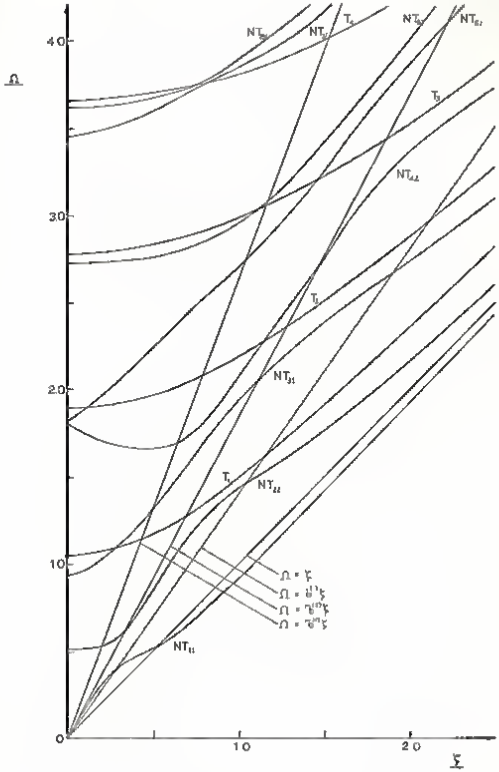

FIGURE 11. Frequency Spectrum: Case 1, H/R = 1.0

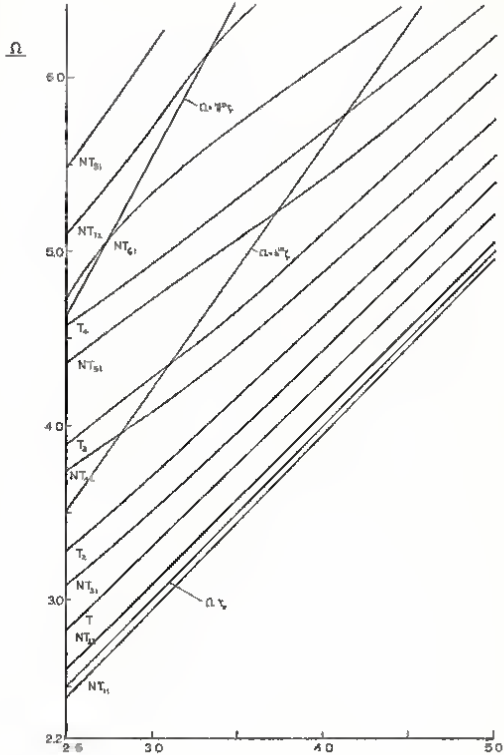

FIGURE 12. Frequency Spectrum: Case I (cont), H/R = 1.0

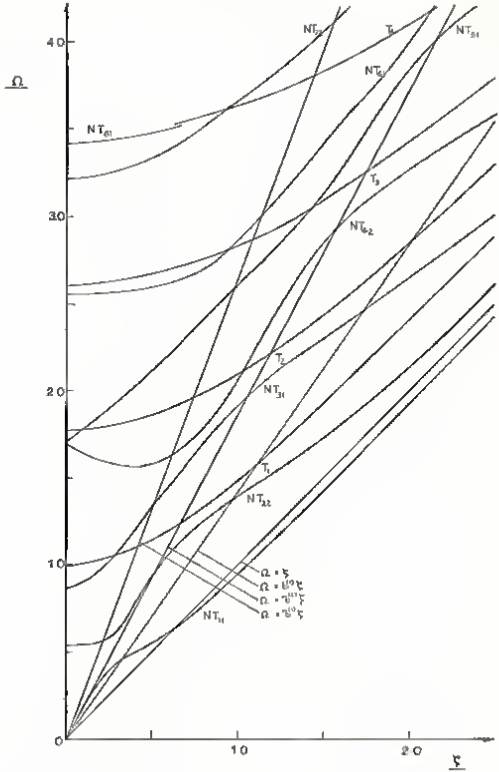

FIGURE 13. Frequency Spectrum: Case II, H/R = 1.0

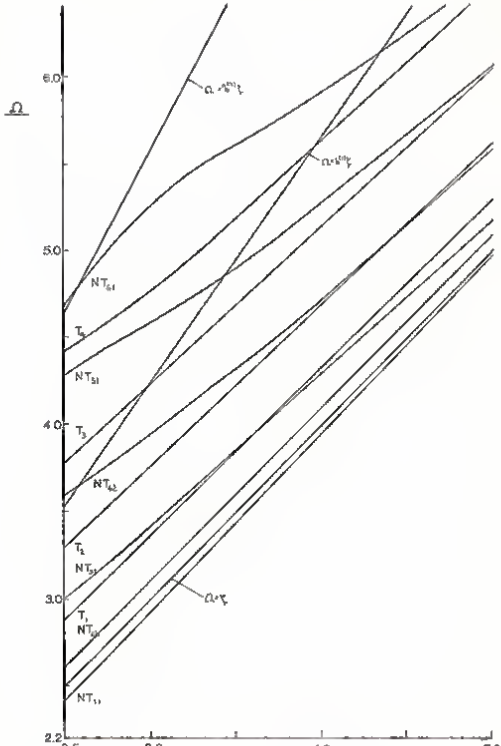

FIGURE 14. Frequency Spectrum: Case II (cont), H/R = 1.0

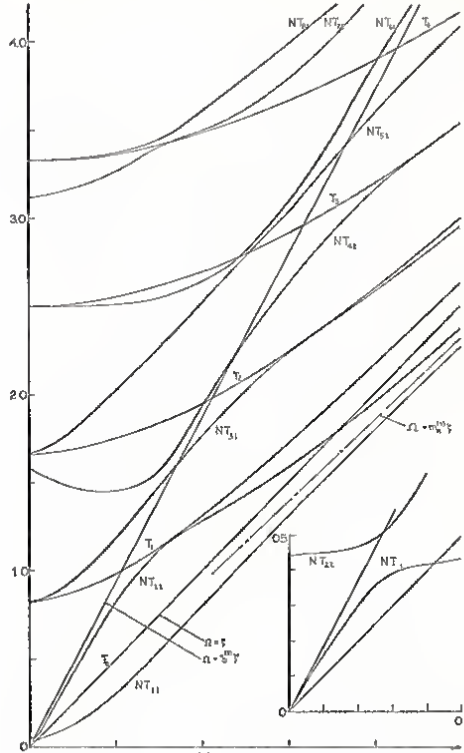

FIGURE 15. Frequency Spectrum: Homogen., H/R = 0.1

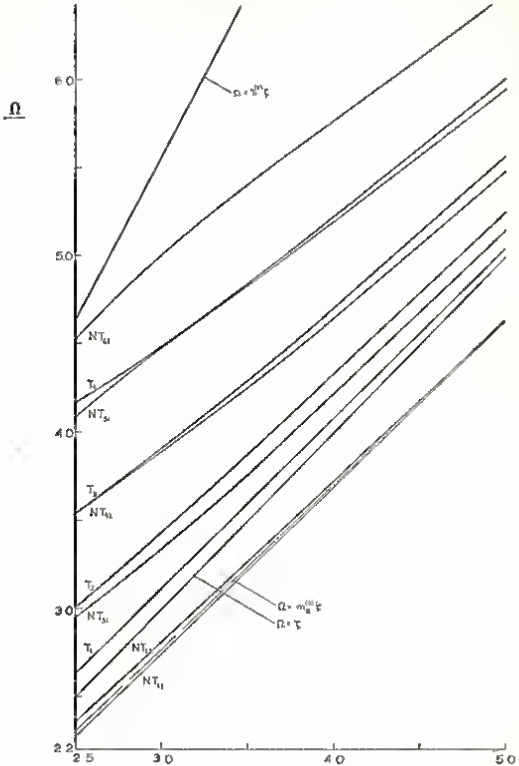

FIGURE 16. Frequency Spectrum: Homogen. (cont), H/R = 0.1

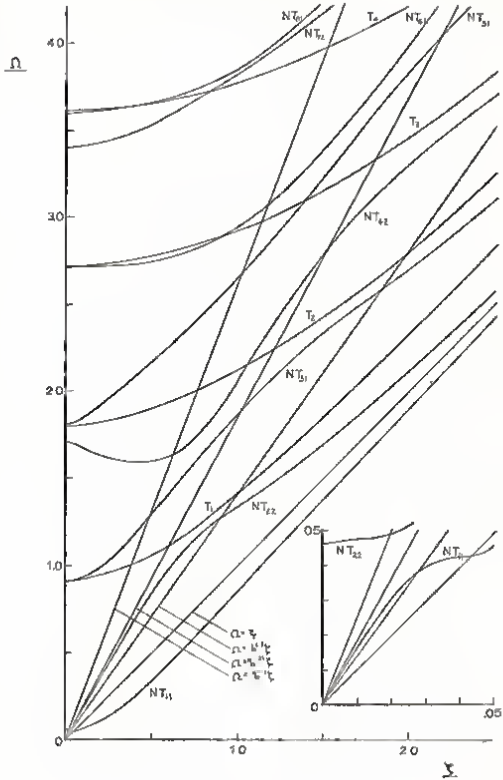

FIGURE 17. Frequency Spectrum: Case I, H/R = 0.1

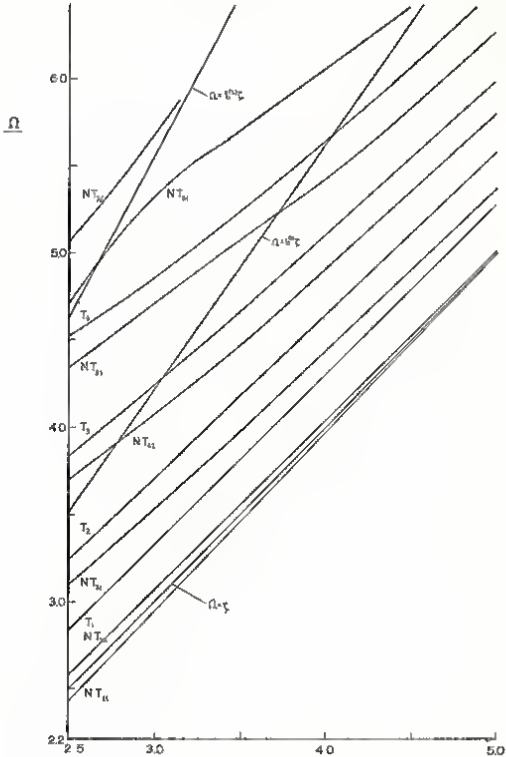

FIGURE 18. Frequency Spectrum: Case I (cont), H/R = 0.1

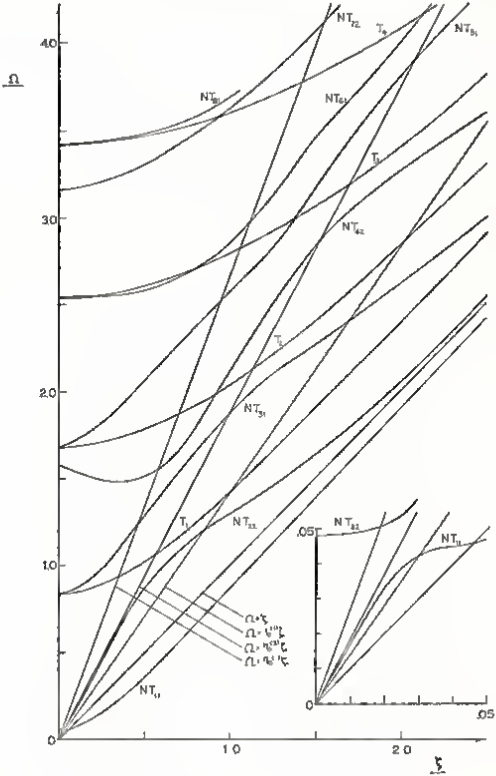

FIGURE 19. Frequency Spectrum: Case II, H/R = 0.1

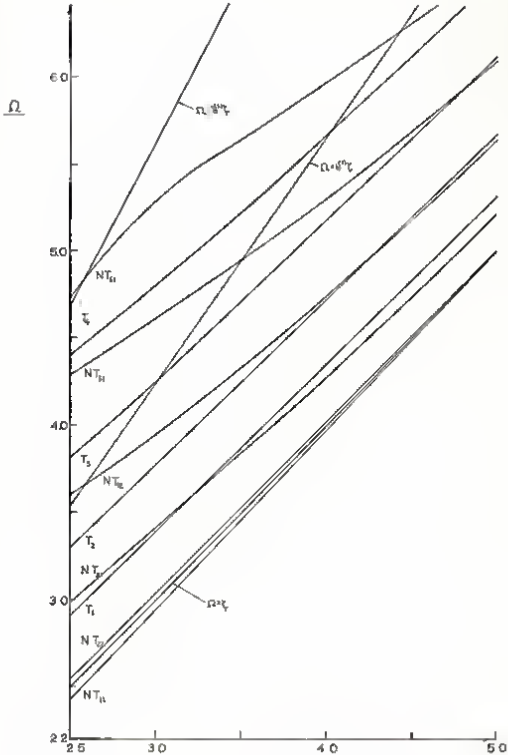

FIGURE 20. Frequency Spectrum: Case II (cont), H/R = 0.1

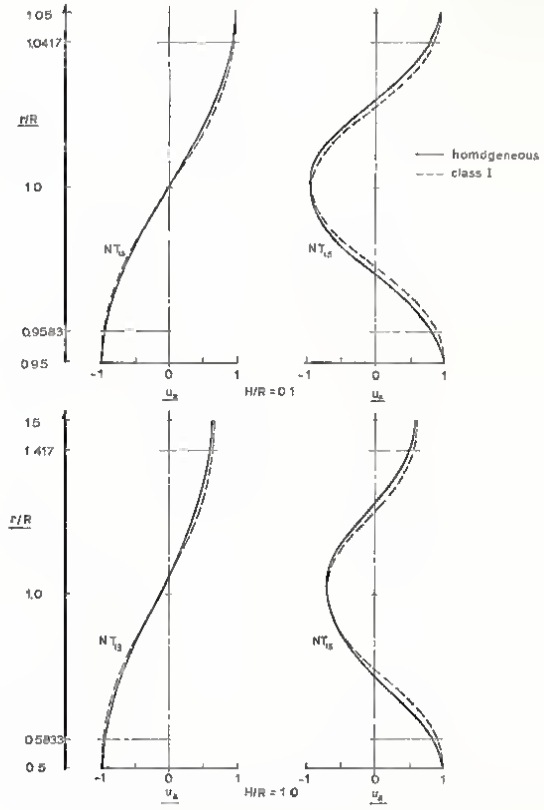

FIGURE 21. Longitudinal-Shear Vibration Distributions

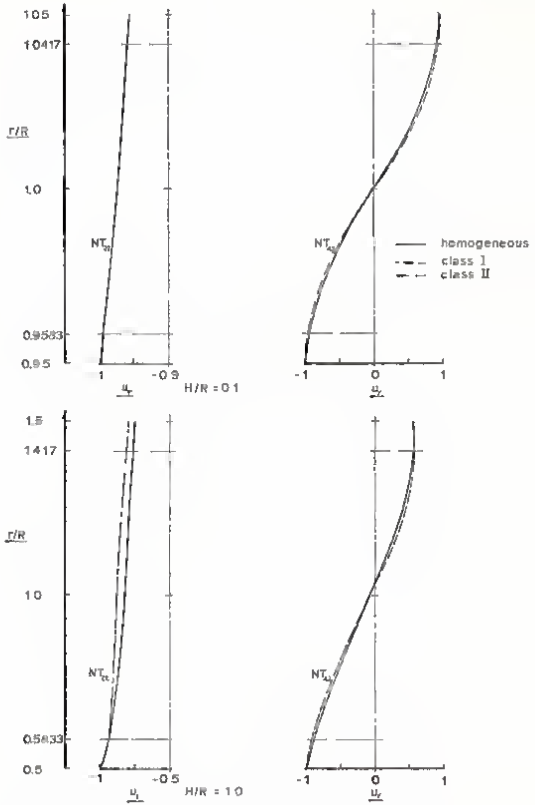

FIGURE 22. Radial Plane-Strain Vibration Distributions

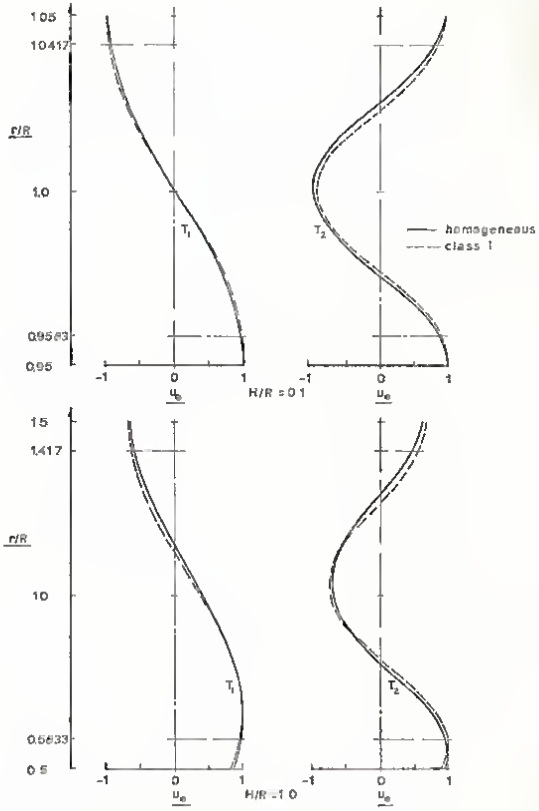

FIGURE 23. Circumferential-Shear Vibration Distributions

BIBLIOGRAPHY

1. J.W.S. Reylaigh, The Theory of Sound, 2nd ed., Dover Publications, Inc., New York, 1945.

2. R.D. Mindlin, An Introduction to the Mathematical Theory of Vibrations of Elastic Plates, U. S. Army Signal Corps Project 1428, 1955.

3. H. Kolsky, Stress Waves in Solids, Dover Publications, Inc., New York, 1963.

4. H.N. Abramson, H.J. Plass, and E.A. Ripperger, "Stress Wave Propagation in Rods and Beams," Advances in Applied Mechanics, Academic Press, New York, 1958.

5. J. Miklowitz, "Recent Developments in Elastic Wave Propagation," Appl. Mech. Rev., Vol. 13, 1960, p. 865.

6. W.L. Pilant, The Propagation of Elastic Waves in Thin Plates, Ph.D. Thesis, Univ. Calif., Los Angeles, 1960.

7. H. Lamb, "On Waves in an Elastic Plate," Proc. Roy. Soc., London, Series A, Vol. 93, 1917, pp. 114-128.

8. L. Pochhammer, "Uebar die Fortpflanzungsgeschwindigkeiten Kleiner Schwingungen in Einem Unbegrenzten Isotropan Kreiscylinder," J. Rein. Angew. Math., Vol. 81, 1876, pp. 324-336.

9. C. Chree, "The Equations of an Isotropic Solid in Polar Cylindrical Coordinates," Trans. Camb. Phil. Soc., Vol. 17, 1889, p. 250.

10. I. Mirsky and G. Herrmann, "Three-Dimensional and Shell-Theory Analysis of Axially Symmetric Motions of Cylinders," J. Appl. Mech., Vol. 23, 1956, pp. 563-568.

11. I. Mirsky and G. Herrmann, "Non-Axially Symmetric Motions of Cylindrical Shells," J. Acous. Soc. Am., Vol. 29, 1957, pp. 1116-1123, and Vol. 31, 1959, p. 250.

12. H.D. McNiven, A.H. Shah, and J.L. Sackman, "Axially Symmetric Waves in Hollow, Elastic Rods: Part I," J. Acous. Soc. Am., Vol. 40, 1966, pp. 784-792.

13. H.D. McNiven, J.L. Sackman, and A.H. Shah, "Axially Symmetric Waves in Hollow, Elastic Rods: Part II," J. Accus. Soc. Am., Vol. 40, 1966, pp. 1073-1076.

14. R.D. Mindlin and H.D. McNiven, "Axially Symmetric Waves in Elastic Rods," J. Appl. Mech., Vol. 27, 1960, pp. 145-151.

15. F.M. Naghdi and R.M. Cooper, "Propagation of Elastic Waves in Cylindrical Shells, Including the Effects of Transverse Shear and Rotatory Inertia," J. Acous. Soc. Am., Vol. 28, 1956, pp. 56-63.

16. P.M. Naghdi and R.M. Cooper, "Propagation of Nonaxially Symmetric Waves in Elastic Cylindrical Shells," J. Acous. Soc. Am., Vol. 29, 1957, pp. 1365-1373.

17. A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed., Dover Publications, Inc., New York, 1927.

18. L.H. Donnell, "Stability of Thin-Walled Tubes Under Torsion," NACA Tech. Rept. No. 479, 1933.

19. R.D. Mindlin, M. Onoe, and H.D. McNiven, "Dispersion of Axially Symmetric Waves in Elastic Rods," J. Appl. Mech., Vol. 29, 1962, pp. 729-734.

20. R.D. Mindlin and Y.H. Pao, "Dispersion of Flexural Waves in an Elastic, Circular Cylinder," J. Appl. Mech., Vol. 27, 1960, p. 513.

21. D.C. Gazis, "A Three-Dimensional Investigation of the Propagation of Waves in Hollow Circular Cylinders - Part I: Analytical Foundation, and Part II: Numerical Results," J. Acous. Soc. Am., Vol. 31, 1960, pp. 568-578.

22. D.C. Gazis, "Exact Analysis of the Plane-Strain Vibrations of Thick-Walled Hollow Cylinders," J. Acous. Soc. Am., Vol. 30, 1958, pp. 786-794.

23. J.E. Greenspon, "Flexural Vibrations of a Thick-Walled Hollow Cylinder," Proceedings of the Third U. S. National Congress on Applied Mechanics (ASME), 1958, pp. 163-173.

24. H.D. McNiven, J.L. Sackman, and A.H. Shah, "Dispersion of Axially Symmetric Waves in Composite, Elastic Rods," J. Acous. Soc. Am., Vol. 35, 1963, pp. 1602-1609.

25. A.E. Armenàkas, "Torsional Waves in Composite Rods," J. Acous. Soc. Am., Vol. 38, 1965, pp. 439-446.

26. J.H. Baltrukonis, W.G. Gottenberg, and R.N. Schreiner, "Dynamics of a Hollow, Elastic Cylinder Contained By an Infinitely Long, Rigid, Circular-Cylindrical Tank," J. Acous. Soc. Am., Vol. 32, 1960, pp. 1539-1546.

27. J.H. Baltrukonis, W.G. Gottenberg, and R.N. Schreiner, "Axial-Shear Vibrations in an Infinitely Long Composite Circular Cylinder," J. Acous. Soc. Am., Vol. 33, 1961, pp. 1447-1457.

28. J.H. Baltrukonis, M. Chi, and W.G. Gottenberg, "Free Transverse Vibrations in an Infinitely Long, Layered, Elastic Cylinder," Catholic University of America, TR 3, 1962.

29. A.E. Armenàkas, "Propagation of Harmonic Waves in Composite Circular Cylindrical Shells. 1: Theoretical Investigation," AIAA Journal, Vol. 5, 1967, pp. 740-744.

30. A.E. Armenàkas and H.E. Keck, "On the Behavior of the Frequency Equation for Composite Rods When the Wave Number is Very Large," to be published.

31. J.P. Jones and J.S. Whittier, "Axially Symmetric Wave Propagation in a Two-Layered Cylinder," Aerospace Corp., El Segundo, Calif., report prepared under U. S. Air Force contract no. AF 04(695)-469.

32. C. Chree, "On the Longitudinal Vibrations of Aelotropic Bars with One Axis of Material Symmetry," Quart. J. Math., Vol. 24, 1890, pp. 340-358.

33. R.W. Morse, "Compressional Waves Along an Anisotropic Circular Cylinder Having Hexagonal Symmetry," J. Acous. Soc. Am., Vol. 26, 1954, pp. 1016-1021.

34. I. Mirsky, "Axisymmetric Vibrations of Orthotropic Cylinders," J. Acous. Soc. Am., Vol. 36, 1964, pp. 2106-2112.

35. I. Mirsky, "Three-Dimensional and Shell-Theory Analysis for Axisymmetric Vibrations of Orthotropic Shells," J. Acous. Soc. Am., Vol. 39, 1966, pp. 549-555.

36. I. Mirsky, "Vibrations of Orthotropic, Thick, Cylindrical Shells," J. Acous. Soc. Am., Vol. 36, 1964, pp. 41-51.

37. J.L. Nowinski, "Propagation of Longitudinal Waves in Circular Cylindrically Orthotropic Bars," J. of Eng. for Ind., (ASME), 1966, pp. 1-5.

38. I. Mirsky, "Elasticity Theory for Anisotropic Media," Rocketdyne, Division of North American Aviation, Inc., Department of Army Project No. 20010501B700.

39. I. Mirsky, "Wave Propagation in Transversely Isotropic Circular Cylinders, Part I: Theory," J. Acous. Soc. Am., Vol. 37, 1965, pp. 1016-1021.

40. I. Mirsky, "Propagation in Transversely Isotropic Circular Cylinders, Part II: Numerical Results," J. Acous. Soc. Am., Vol. 37, 1965, pp. 1022-1026.

41. Y.Y. Yu, "Vibrations of Elastic Sandwich Cylindrical Shells," J. Appl. Mech., Vol 27, 1960, pp. 653-662.

42. H. Chu, "Vibrations of Honeycomb Sandwich Cylinders," J. Aero. Sci., Vol. 28, 1961, pp. 930-939.

43. J.D. Achenbach and G. Herrmann, "Dispersion of Free Harmonic Waves in Fiber-Reinforced Composites," Northwestern University, Evanston, Ill., Office of Naval Research Contract No. 1228(34), 1967.

44. R.F.S. Hearmon, An Introduction to Applied Anisotropic Elasticity, Oxford, at the Clarendon Press, 1961.

45. L.H. Van Vlack, Elements of Materials Science, Addison-Wesley, Inc., Reading, Mass., 1960.

46. R.W. Fenn, D.D. Crooks, W.C. Kinder, and F.M. Lempriere, "Test Methods for Mechanical Properties of Anisotropic Materials (Beryllium Sheet)," Lockheed Missiles and Space Company Tech. Rept. No. AFML-TR-67-212, 1967.

47. Y.C. Fung, Foundations of Solid Mechanics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965.

48. I.S. Sokolnikoff, Mathematical Theory of Elasticity, 2nd ed., McGraw-Hill, Inc., New York, 1956.

49. W.V. Parker and J.C. Eaves, Matrices, Ronald Press Co., New York, 1960.

50. S.D. Poisson, "Memoire sur L'equilibre it le Mouvement des Corps Elastiques," Mem. Acac. Sci., Paris, Vol. 8, 1829, p. 623.

51. G. Lamé, Lecons sur la Theorie Mathematique de L'elasticite des Corps Solides, Paris, Bachelier, 1852.

52. A. Clebsh, "Uber die Reflexion an einer Kugelflache," J. Rein. Angew. Math., Vol. 61, 1863, p. 195.

53. C. Somigliana, "Sulle Espressioni Analitiche Generali dei Movimenti Oscillatori," Atti Reale Accard. Linc., Roma, Vol. 5, 1892, p. 111.

54. O. Tedone, "Sulle Vibrazioni dei Corpi Solidi, Omogenei ed Isotropi," Mem. Reale Accad, Scienze Torino, Vol. 47, 1897, p. 181.

55. P. Duhem, "Sur L'integrale des Equations des Petits Mouvements d'un Solide Isotrope," Mem. Soc. Sci., Bordeaux, Vol. 3, 1898, p. 316.

56. E. Sternberg, "On the Integration of the Equations of Motion in the Classical Theory of Elasticity," Arch. Rat. Mech. Anal., Vol. 6, 1959, pp. 34-50.

57. S.G. Lekhnitsky, "Theory of Elasticity of an Anisotropic Body," (in Russian), Gosudarstv. Izdad. Tehn. Teor. Lit., Moscow-Leningrad, 1950.

58. H.A. Elliot, "Three-Dimensional Stress Distributions in Hexagonal Aeolotropic Crystals," Proc. Camb. Phil. Soc., Vol. 44, 1948, p. 522.

59. R.A. Eubanks and E. Sternberg, "On the Axisymmetric Problem of Elasticity Theory for a Medium with Transverse Isotropy," J. Rat. Mech. Anal., Vol. 3, 1954, pp. 89-101.

60. M.J.P. Musgrave, "On the Propagation of Elastic Waves in Aeolotropic Media 1: General Principles," Proc. Roy. Soc., Vol. A226, 1954, p. 339.

61. F.B. Hildebrand, Advanced Calculus for Applications, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1st Revision, 1962.

62. E.L. Ince, Ordinary Differential Equations, Dover Publications, Inc., New York, 1956.

63. E. Almansi, "Sull' Integrazione dell' Equazione Differenziale $\Delta\Delta\Delta u = 0$," Ann. di Mat., Vol. 2, 1899, p. 1.

64. C.E. Weatherburn, Advanced Vector Analysis, Open Court Publishing Co., LaSalle, Ill., 1947.

65. P.M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill Book Company, Inc., New York, 1953.

66. G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 2nd ed., 1966.

67. J. Oliver, "Elastic Wave Dispersion in a Cylindrical Rod by a Wide-Band Short-Duration Pulse Technique," J. Acous. Soc. Am., Vol. 29, 1957, p. 189.

68. M. Onoe, "Contour Vibration of Thin Rectangular Plates," J. Acous. Soc. Am., Vol. 30, 1958, p. 1159.

69. J. Zemanek, An Experimental and Theoretical Investigation of Elastic Wave Propagation in a Cylinder, Ph.D. Thesis, University of Calif., Los Angeles, 1962.

70. E.A.G. Shaw, "On the Resonant Frequencies of Thick Barium Titanate Disks," J. Acous. Soc. Am., Vol. 28, 1956, p. 38.

71. C.S. Yih, "Solution of the Hyper-Bessel Equation," Quart. Appl. Math., Vol. 13, 1956, pp. 462-463.

72. R.D. Mindlin, "Waves and Vibrations in Isotropic, Elastic Plates," Proceedings of the First Symposium on Naval Structural Mechanics, Pergamon Press, 1960, pp. 199-232.

73. M.J.P. Musgrave, "On the Propagation of Elastic Waves in Aeolotropic Media II: Media of Hexagonal Symmetry," Proc. Roy. Soc., Vol. A226, 1954, p. 356.

74. Sandwich Construction for Aircraft, ANC-23, 2nd ed., Department of the Air Force ARDC, 1955.

75. H. Chu, "Non-Axisymmetric Vibrations of Honeycomb Sandwich Cylinders," Martin-Denver Research Report No. R-60-28, 1960.

76. Y.Y. Yu, "Flexural Vibrations of Elastic Sandwich Plates," J. Aero/Space Sci., Vol. 27, 1960, pp. 271-282.

77. Y.Y. Yu, "A New Theory of Elastic Sandwich Plates - One Dimensional Case," J. Appl. Mech., Vol. 26, 1959, pp. 415-421.

78. J.H. Baltrukonis and W.G. Gottenberg, "Thickness-Shear Vibrations in Circular Bars," J. Acous. Soc. Am., Vol. 31, 1959, pp. 734-739.

79. W.M. Ewing, W.S. Jardetzky, and F. Press, Elastic Waves in Layered Media, McGraw-Hill Book Company, Inc., New York, 1957.

80. L. Knopoff, "On Rayleigh Wave Velocities," Bull. Seism. Soc. Am., Vol. 42, 1952, pp. 307-308.

81. R. Stoneley, "Elastic Waves at the Surface of Separation of Two Solids," Proc. Roy. Soc., Vol. A106, 1924, pp. 416-428.

82. J.G. Scholte, "On the Stoneley-Wave Equation I," Proc. Ned. Akad. v. Wetensch., Vol. 45, 1942, pp. 20-25.

83. J.G. Scholte, "On the Stoneley-Wave Equation II," Proc. Nad. Akad. v. Watensch., Vol. 45, 1942, pp. 159-164.

84. J.G. Scholte, "On Surface Waves in a Stratisfied Medium," Proc. Nad. Akad. v. Watensch., Vol. 45, 1942, pp. 380-386.

85. J.G. Scholte, "On Surface Waves in a Stratisfied Medium," Proc. Nad. Akad. v. Watensch., Vol. 45, 1942, pp. 449-456.

86. J.G. Scholte, "On Surface Waves in a Stratisfied Medium," Proc. Nad. Akad. v. Wetensch., Vol. 45, 1942, pp. 516-523.

87. J.G. Scholte, "The Range of Existence of Rayleigh and Stoneley Waves," Monthly Notices Roy. Astron. Soc.: Geophys. Suppl., Vol. 5, 1947, pp. 120-126.

88. H. Koppe, "Uber Rayleigh-Wellen an der Oberflache zweier Medien," Zait. Angew. Math. und Mech., Vol. 28, 1948, pp. 355-360.

89. K. Sazawa and K. Kanai, "The Range of Possible Existence of Stoneley-Waves," Bull. Earthq. Res. Inst., Vol. 17, 1938, pp. 1-8.

90. K. Sazawa and K. Kanai, "Discontinuity in the Dispersion Curves of Rayleigh Waves," Bull. Earthq. Res. Inst., Vol. 13, 1935, pp. 237-244.

91. K. Sazawa and K. Kanai, "Anomalous Dispersion of Elastic Surface Waves. I," Bull. Earthq. Res. Inst., Vol. 16, 1938, pp. 225-233.

92. K. Sazawa and K. Kanai, "Anomalous Dispersion of Elastic Surface Waves. II," Bull. Earthq. Res. Inst., Vol. 16, 1938, pp. 683-689.

93. A.E.H. Love, Some Problems of Geodynamics, Cambridge University Press, 1911.

94. T.R. Meeker and A.H. Meitzler, "Guided Wave Propagation in Enlongated Cylinders and Plates," Physical Acoustics, Part IA, Academic Press, Inc., 1964.

95. A.E. Armenàkas, "On the Accuracy of Some Dynamic Shell Theories," Proc. Am. Soc. Civil Eng., Vol. EM5, 1967, pp. 95-109.

96. V.Z. Vlasov, General Theory of Shells and Its Applications in Engineering, NASA Technical Translation No. TTF-99, translated 1964.

BIOGRAPHICAL SKETCH

Henry Evans Keck was born on November 29, 1936, in San Pedro, California. From the age of two, he was reared in Virginia. He attended high school in Fluvanna County and was graduated in June, 1955.

In October, 1956, Henry joined the United States Air Force and, after technical schooling, was stationed at Hunter Air Force Base at Savannah, Georgia. He attended night classes at Armstrong Junior College, and in 1961 the Air Force accepted his application to attend college on a full time basis. He was sent to the University of Arizona, where, in 1963 he was graduated with highest distinction with a Bachelor of Science Degree in Mechanical Engineering.

After Officer Training School at Lackland Air Force Base, San Antonio, Texas, the new Second Lieutenant Keck was stationed at Cape Kennedy, where from October, 1963, until July, 1967, he was a Project Officer on the Air Force Titan III space launch vehicle. In the period from September, 1964, until May, 1966, Lieutenant Keck attended night classes in the GENESYS (Graduate Engineering Education System) system of the University of Florida and obtained a Master of Science Degree with major in

Engineering Science and Mechanics and minor in Mathematics.

In the spring of 1966, Lieutenant Keck was again selected by the Air Force to attend college on a full time basis and, since September 1966, has been pursuing the degree of Doctor of Philosophy with major in Engineering Science and Mechanics and minor in Mathematics.

This dissertation was prepared under the direction of the dissertation chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1968

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee:

Chairman